II YEAR–III SEMESTER
COURSE CODE: 7MMA3C1

CORE COURSE-IX–COMPLEX ANALYSIS

Unit I

Concept of analytic function – Elementary theory of power series – Conformability – Linear transformations.

Unit II

Complex integration – Cauchy integral formula.

Unit III

Local properties of analytic functions.

Unit IV

Calculus of residues.

Unit V

Power series expansions – canonical products – Jensen's formula.

Text Book

Lars V.Ahlfors, Complex Analysis, 3rd edition, McGraw Hill International Book Company, 1979.

Chapter II	:	(Sections 1, 2)
Chapter III	:	(Sections 2, 3)
Chapter IV	:	(Sections 1, 2, 3, & 5)
Chapter V	:	(Sections 1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 3.3).

Books for Supplementary Reading and Reference:

1. S.Ponnusamy, Foundations of Complex Analysis, Narosa Publication House, New Delhi, 2004.
2. John B.Conway, Functions of One Complex Variable, 2nd edition, Springer-Verlag, International Student Edition, Narosa Publishing Company.

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UNIT - I

* Concept of analytic function
* Elementary theory of power series
* conformability.
* Linear transformations

Unit - I

Lucas theorem

Statement:

If all zeros of a polynomial $p(z)$ lie in a half plane. Then all zeros (roots) of the derivatives $p'(z)$ lie in the same half plane

proof:- Let $P(z) = (z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$ where α_i are not necessarily distinct. The hypothesis implies that

$$Im\left[\frac{(\alpha_i - a)}{b}\right] < 0 \quad \text{for } 1 \le i \le n$$

If z is not in the half plane. in which α_j's lie. Then

$$Im\left(\frac{z-a}{b}\right) \ge 0 \text{ for any such } z'$$

we have,

$$p'(z) = (z-\alpha_2)(z-\alpha_3)\ldots(z-\alpha_n) + (z-\alpha_1)(z-\alpha_3)\ldots(z-\alpha_n) + \ldots + (z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{n-1})$$

$$\frac{p'(z)}{p(z)} = \frac{1}{z-\alpha_1} + \frac{1}{z-\alpha_2} + \ldots + \frac{1}{z-\alpha_n}$$

$$= \sum_{i=1}^{n} \frac{1}{z-\alpha_i} \text{ and hence}$$

$$Im\left(\frac{z-\alpha_i}{b}\right) = Im\left(\frac{z-a}{b}\right) - Im\left(\frac{\alpha_i - a}{b}\right) > 0$$

But the imaginary parts of reciprocal numbers have opposite sign and at conclude the $Im\left(\frac{b}{z-\alpha_i}\right) < 0 \quad (1 \le i \le n)$

Hence $Im\left(b\,\frac{p'(z)}{p(z)}\right) = \sum_{i=1}^{n} Im\left(\frac{b}{z-\alpha_i}\right) < 0$

and hence $p'(z) \neq 0$

We have proved that,

If z is not in the same half plane as α's

Then $P'(z) \neq 0$

Thus all the zeroes of $P'(z)$ lie in the same half plane

Hence proved.

Corollary:-

The smallest convex polygon that contains all the zeros of $P(z)$ also contains all the ~~roots~~ zeroes of $P'(z)$.

Abel theorem:

For every power series $\sum_{n=0}^{\infty} a_n z^n$ there exists a number R, $0 \leq R \leq \infty$ called the radius of convergence is the following properties:

i) The series converges absolutely for every z with $|z| < R$. If $0 \leq \rho < R$, the converges is uniform for $|z| \leq \rho$

ii) If $|z| > R$, the terms of a series are unbounded and series is consequently divergence.

iii) If $|z| < R$. Thus the sum of the series is analytic function. The derivative can be obtain by termwise differentiation and the derive series has the same radius of convergence.

Proof:-

Let R be a radius of convergence

$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$

i) If $|z| < R$

we can find $\rho : |z| < \rho < R$

Then $\frac{1}{\rho} > \frac{1}{R}$ by defn of supremum

there exists an n_0 such that $|a_n|^{1/n} < 1/\rho$

$\Rightarrow |a_n| < \frac{1}{\rho^n}$ for $n \geq n_0$

$\Rightarrow |a_n z^n| < \frac{z^n}{\rho^n}$

$\Rightarrow \sum |a_n z^n| < \sum \left(\frac{z}{\rho}\right)^n$

which is the geometric series

$\therefore \sum_{n=0}^{\infty} a_n z^n$ has a convergent geometric series

and is consequently convergent.

$\therefore \sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent.

Next to prove: uniform convergence.

$|z| < \rho < R$

We choose ρ' with $\rho < \rho' < R$

The series $\sum a_n z^n$ is bounded.

$\sum |a_n z^n| \leq \sum |a_n [\rho']^n| \leq K$ — (1)

$\leq \sum \left| a_n (\rho')^n \frac{z^n}{(\rho')^n} \right|$

$= \sum |a_n (\rho')^n| \, |(z/\rho')^n|$

$= \sum |a_n (\rho')^n| \, |(z/\rho')^n|$

$= \sum |K| \left|\frac{z}{\rho'}\right|^n$

$= |K| \sum \frac{|z|^n}{|\rho'|^n}$

$$\leq |K| \sum \frac{\rho^n}{\rho'^n}$$

$$= |K| \sum \left(\frac{\rho}{\rho'}\right)^n$$

By weistrass m-test $\sum a_n z^n$ is uniformly convergent.

ii) If $|z| > R$

we choose ρ so that $R < \rho < |z|$.

Since $\frac{1}{\rho} < \frac{1}{R}$ for large n there exists

$$|a_n|^{1/n} > 1/\rho$$

$$\Rightarrow |a_n| > 1/\rho^n$$

Thus $\sum |a_n z^n| > \sum (z/\rho)^n$ for infinitely many n.

$\sum a_n z^n$ is unbounded and hence it is divergence

iii) Let $\sqrt[n]{n} = 1 + \delta_n \quad , \delta_n > 0$

$$n = (1+\delta_n)^n > 1 + \tfrac{1}{2}(n(n-1))\delta_n^2$$

$$\Rightarrow \frac{n(n-1)}{2} \delta_n^2 < n-1$$

$$\Rightarrow n\delta_n^2 < 1(2)$$

$$\Rightarrow \delta_n^2 < 2/n \text{ and hence } \delta_n \to 0$$

for $|z| < R$

$$\text{Let } f(z) = \sum_{n=0}^{\infty} a_n z^n = S_n(z) + R_n(z)$$

where $S_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1}$

$$= \sum_{n=0}^{n-1} a_K z^n$$

$$R_n(z) = \sum_{K=n}^{\infty} a_K z^K$$

and also $S_n'(z) = \sum_{n=1}^{n-1} a_n n z^{n-1}$

Also, $f_1(z) = \sum_{1}^{\infty} n a_n z^{n-1}$

$$f_1(z) = \lim_{n\to\infty} S_n'(z)$$

We have to show that $f'(z) = f_1(z)$

consider the identity,

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) = \frac{S_n(z)+R_n(z)-R_n(z_0) - S_n(z_0)}{z-z_0} - f_1(z_0)+S_n'(z_0)-S_n'(z_0)$$

$$= \left[\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right] + [S_n'(z_0)-f_1(z_0)] + \left[\frac{R_n(z)-R_n(z_0)}{z-z_0}\right] \quad \text{—①}$$

$$\frac{R_n(z)-R_n(z_0)}{z-z_0} = \frac{\sum_{k=n}^{\infty} a_k z^k - \sum_{k=n}^{\infty} a_k z_0^k}{z-z_0}$$

$$= \sum_{k=n}^{\infty} \frac{a_k(z^k - z_0^k)}{z-z_0}$$

$$= \sum_{k=n}^{\infty} \frac{a_k}{z}(z^k - z_0^k)\left(1-\frac{z_0}{z}\right)^{-1}$$

$$= \sum_{k=n}^{\infty} a_k(z^{k-1} - z_0^k z^{-1})\left[1+\frac{z_0}{z}+\frac{z_0^2}{z^2}+\cdots+\left(\frac{z_0}{z}\right)^{k-1}+\left(\frac{z_0}{z}\right)^k+\left(\frac{z_0}{z}\right)^{k+1}\cdots\right]$$

where $z \neq z_0$ and $|z|, |z_0| < \rho < R$
(In paper)

Let R_1 be the series of convergences of $\sum_1^{\infty} n a_n z^{n-1}$

$$\frac{1}{R_1} = \lim_{n\to\infty} \sup \sqrt[n]{|na_n|}$$

from ⊛ $\sqrt[n]{n} = 1$ as $n \to \infty$

$$\frac{1}{R_1} = \lim_{n\to\infty} \sup \sqrt[n]{n|a_n|}$$

$$= \lim_{n\to\infty} \sup \sqrt[n]{n}\,\sqrt[n]{|a_n|}$$

$$= \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$$

$= 1/R$

Hence the radius of convergence as same

Differentiable curve:

A curve γ is said to be differentiable if $z'(t)$ exists and is continuous.

Regular or smooth curve:

A curve is said to be regular or smooth if $z'(t) \neq 0$

pointwise differentiable curve: (piecewise)

A curve c is given by $z = z(t)$ is said to piecewise differentiable if it is differentiable except a finite number of points and at any point where $z(t)$ is not differentiable it has a left derivative and right derivative.

conformal:

Let f be a continuous function define in the region D. Let $z_0 \in D$. Let c_1 and c_2 be two regular curve passing through z_0 and lying in D.

Let c_1' and c_2' be the image of c_1 and c_2 respectively under 'f' if the angle between c_1 and c_2 is equal to the angle between c_1' and c_2' both in magnitude and direction. Then 'f' is said to be conformal at z_0.

Isogonal:

If the angle is preserved only in magnitude and direction is reversed. Then the mapping is said to be isogonal or indirectly conformal.

Abel limit thm / II part of Abel's thm

Statement:

If $\sum_0^\infty a_n$ converges, then $f(z) = \sum_0^\infty a_n z^n$ tends to $f[1]$ as approaches 1 in such a way that $\frac{1-z}{1-|z|}$ remains bounded.

proof:- We may assume that $\sum_0^\infty a_n = 0$ for this can be attained by adding a constant to a_0

We write,

$s_n = a_0 + a_1 + \dots + a_n$ and we make of identity

$$s_n(z) = a_0 + a_1 z + \dots + a_n z^n$$

$$= s_0 + (s_1 - s_0)z + \dots + (s_n - s_{n-1})z^n$$

where $n=0$ in $s_n \Rightarrow s_0 = a_0$

$n=1$ in $s_n \Rightarrow s_1 = a_0 + a_1$

$\Rightarrow s_1 = s_0 + a_1$

$\Rightarrow a_1 = (s_1 - s_0)$

$\Rightarrow a_2 = (s_2 - s_1)$

Similarly $n = n-1$ in $s_n \Rightarrow$

$s_{n-1} = a_0 + a_1 + \dots + a_{n-1}$

$s_{n-1} = s_{n-2} + a_{n-1}$

$\Rightarrow a_{n-1} = (s_{n-1} - s_{n-2})$

$n=n$ in $s_n \Rightarrow s_n = a_0 + a_1 + \dots + a_n$

$s_n = s_{n-1} + a_n$

$a_n = (s_n - s_{n-1})$

$\Rightarrow S_n(z) = S_0 + (S_1 - S_0)z + \dots + (S_n - S_{n-1})z^n$

$= S_0(1-z) + S_1(z-z^2) + \dots +$

$S_{n-1}(z^{n-1} - z^n) + S_n z^n$

$= S_0(1-z) + S_1 z(1-z) + S_2 z^2(1-z) +$

$\dots + S_{n-1} z^{n-1}(1-z) + S_n z^n$

$S_n(z) = (1-z)\left[S_0 + S_1 z + S_2 z^2 + \dots + S_{n-1} z^{n-1}\right] + S_n z^n$

$S_n(z) = (1-z)\left[S_0 + S_1 z + S_2 z^2 + \dots + S_{n-1} z^{n-1}\right] + S_n z^n$

But $S_n z^n \rightarrow 0$ we obtain the representation

$|f(z)| = |(1-z) \sum_0^\infty S_n z^n|$

$\leq |1-z| \left[|\sum_{n=0}^{m-1} S_n z^n| + |\sum_{n=m}^{\infty} S_n z^n| \right]$

$\leq |1-z| |\sum_{n=0}^{m-1} S_n z^n| + |1-z| |\sum_{n=m}^{\infty} S_n z^n|$

$< |1-z| \sum_{n=0}^{m-1} |S_n z^n| + |1-z| \sum_{n=m}^{\infty} |S_n z^n|$

$< |1-z| \sum_{n=0}^{m-1} |S_n z^n| + |1-z| \, \varepsilon \frac{|z|^m}{1-|z|}$

$\left[\sum_{n=m}^{\infty} |z|^n = |z|^m + |z|^{m+1} + \dots \right.$

$= |z|^m \left[1 + |z| + |z|^2 + \dots\right]$

$= |z|^m \left[\frac{1}{1-|z|}\right]$

$\therefore |f(z)| \leq |1-z| \sum_{n=0}^{m-1} |S_n z^n| + K\varepsilon$

The 1^{st} term on the right is arbitrary small by choosing z sufficient close by 1.

We conclude that $f(z) \rightarrow 0$ as $z \rightarrow 1$

pblm 1

Find the radius of convergence of the following power series

i) $\sum \frac{1}{n^n} z^n$ ii) $\sum \frac{z^n}{n!}$

soln:- The given series is $\sum \frac{1}{n^n} z^n$

Here $a_n = \frac{1}{n^n}$

W.K.T, $\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$

$$= \lim_{n\to\infty} \left(\frac{1}{|n^n|}\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) = \frac{1}{\infty} = 0$$

$$\frac{1}{R} = 0 \Rightarrow R = \infty$$

ii) The gvn power series $\sum \frac{z^n}{n!}$

Here $a_n = \frac{1}{n!}$ and $a_{n+1} = \frac{1}{(n+1)!}$

W.K.T, $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$$= \lim_{n\to\infty} \left|\frac{1/n!}{1/(n+1)!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{1}{n!} \times \frac{(n+1)\,n!}{1}\right|$$

$$= \lim_{n\to\infty} (|n+1|)$$

$$R = \infty.$$

pblm: 2 Find the fixed point of the following linear transformation

(i) $w = \frac{z}{2z-1}$ (ii) $w = \frac{2z}{3z-1}$ (iii) $w = \frac{3z-4}{z-1}$

soln:- (i) given $w = \frac{z}{2z-1}$

put $w = z$

$$z = \frac{z}{2z-1}$$

$2z^2 - z - z = 0 \Rightarrow 2z = 0$ (or) $(z-1) = 0$

$2z^2 - 2z = 0 \Rightarrow 2z(z-1) = 0$

$\Rightarrow z = 0$ (or) $z = 1$

$\therefore z = 0, 1$ are fixed point

(ii) Given $w = \frac{2z}{3z-1}$

put $w = z$

$$z = \frac{2z}{3z-1}$$

$$\Rightarrow 3z^2 - z - 2z = 0$$

$$\Rightarrow 3z^2 - 3z = 0$$

$$\Rightarrow 3z(z-1) = 0$$

$$3z = 0 \qquad z-1 = 0$$

$$\Rightarrow z = 0 \qquad z = 1$$

$z = 0, 1$ are fixed points.

iii) Given $w = \frac{3z-4}{z-1}$

put $w = z$

$$z = \frac{3z-4}{z-1}$$

$$\Rightarrow z^2 - z - 3z + 4 = 0$$

$$\Rightarrow z^2 - 4z + 4 = 0$$

$$\Rightarrow (z-2)(z-2) = 0$$

$$\Rightarrow z = 2, 2.$$

$z = 2, 2$ are fixed point.

5) Find the linear transformation which carries $(0, i, -i)$ and $(1, -1, 0)$

Soln:-

Given $z_1 = 0$, $z_2 = i$, $z_3 = -i$

$w_1 = 1$, $w_2 = -i$, $w_3 = 0$

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_3)}$$

$$\frac{(w+1)(1-0)}{(w-0)(1+1)} = \frac{(z-i)(0+i)}{(z+i)(0-i)}$$

$$\frac{w+1}{2w} = \frac{(z-i)\,i}{(z+i)(-i)}$$

$$= \frac{z-i}{-(z+i)}$$

$$-(z+i)(w+1) = (z-i)\,2w$$

$$-wz - iw - z - i = 2zw - 2iw$$

$$-wz - 2zw - wi - z - i + 2iw = 0$$

$$-3zw + wi - z - i = 0$$

$$w(-3z+i) - (z+i) = 0$$

$$w(-3z+i) = z+i$$

$$w = \frac{z+i}{-3z+i}.$$

Find the linear transformation which carries $(-2, 0, 2)$ into $(0, i, -i)$.

Soln:- Given $z_1 = -2,\ z_2 = 0,\ z_3 = 2$

$$w_1 = 0,\quad w_2 = i,\quad w_3 = -i$$

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-0)(-2-2)}{(z-2)(-2-0)}$$

$$\frac{(w-i)(i)}{(w+i)(-i)} = \frac{-4z}{-2(z-2)}$$

$$\frac{(w-i)}{-(w+i)} = \frac{2z}{z-2}$$

$$(w-i)(z-2) = -2z(w+i)$$

$$wz - 2w - iz + 2i = -2zw - 2iz$$

$$wz - 2w - iz + 2i + 2wz + 2iz = 0$$

$$3wz + iz - 2w + 2i = 0$$

$$w(3z-2) = -iz - 2i$$

$$w = \frac{-i(z+2)}{3z-2}.$$

Find a linear transformation which carries $(2, i, -2)$ into $(1, i, -1)$

Soln:- Given $z_1 = 2,\ z_2 = i,\ z_3 = -2$

$$w_1 = 1,\ w_2 = i,\ w_3 = -1$$

$$\frac{(w-i)(1+1)}{(w+1)(1-i)} = \frac{(z-i)(2+2)}{(z+2)(2-i)}$$

$$\frac{(w-i)(2)}{(w+1)(1-i)} = \frac{(z-i)(4)}{(z+2)(2-i)}$$

$$(z+2)(2-i)(w-i) = 2(z-i)(w+1)(1-i)$$

$$(2z-iz+4-2i)(w-i) = 2(zw+z-iw-i)(1-i)$$

$$2zw - izw + 4w - 2iw - 2iz + (-1)z - 4i + (-1)2 = 2zw + 2z - 2iw - 2i - 2izw - 2iz + (-2)w + (-2)$$

$$2zw + 4w - 2iw - 2iz - z - 4i - 2 - 2zw - 2z + 2iw + 2i + 2iz + 2w + 2 + iwz = 0$$

$$6w - 3z - 2i + iwz = 0$$

$$w(6+iz) = 3z + 2i$$

$$w = \frac{3z+2i}{6+iz}.$$

Linear transformation

A linear transformation is $w = S(z) = \dfrac{az+b}{cz+d}$ with $ad-bc \neq 0$ has an inverse $z = S^{-1}(w)$

$$\Rightarrow \frac{dw-b}{-cw+a}$$

Linear fractional transformation (or) Bilinear transformation (or) Mobius transformation.

A transformation of the form $w = \dfrac{az+b}{cw+d}$, $ad-bc \neq 0$ where a, b, c, d are complex numbers called a linear transformation.

cross ratio:

The cross ratio (z_1, z_2, z_3, z_4) is the image of z under the linear transformation which carries (z_2, z_3, z_4) into $(1, 0, \infty)$

Given four points (z_1, z_2, z_3, z_4) taken in order the ratio.

$\dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ is called the cross ratio of the points.

fixed points:

A point which are invarient under a transformation are called fixed point (or) constant point (or) invarient point are of the transformation.

The fixed point of the transformation $w = \dfrac{az+b}{cz+d}$ is given by

$z = \dfrac{az+b}{cz+d}$ (or) $(z^2 + dz - az - b) = 0$.

pblm: 1 $T_1 z = \dfrac{z+2}{z+3}$, $T_2 z = \dfrac{z}{z+1}$. find $T_1^{-1}(z), T_1 T_2(z)$ $T_2 T_1(z)$, $T_1^{-1} T_2(z)$, $T_1^{-1} T_1(z)$

soln:-

Given $T_1 z = \dfrac{z+2}{z+3}$

Let $T_1 z = \dfrac{z+2}{z+3} = w$ (say)

$$\Rightarrow (z+3) w = z+2$$

$$wz + 3w = z + 2$$

$$wz + 3w - z - 2 = 0$$

$$z(w-1) + 3w - 2 = 0$$

$$z = \frac{2-3w}{w-1}$$

i) $T_1^{-1}(w) = \dfrac{2-3w}{w-1}$

ii) $T_1 T_2(z) = T_1(T_2(z)) = T_1\left(\dfrac{z}{z+1}\right)$

$$= \frac{\left(\frac{z}{z+1}\right) + 2}{\left(\frac{z}{z+1}\right) + 3}$$

$$=\left(\frac{z+2z+2}{z+1}\right)\Big/\left(\frac{z+3z+3}{z+1}\right)$$

$$=\frac{z+2z+2}{z+3z+3}$$

$$T_1T_2(z)=\frac{3z+2}{4z+3}$$

iii) $T_2T_1(z)=T_2(T_1(z))$

$$=T_2\left(\frac{z+2}{z+3}\right)$$

$$=\left(\frac{z+2}{z+3}\right)\Big/\left(\frac{z+2}{z+3}+1\right)$$

$$=\frac{\frac{z+2}{z+3}}{\frac{2z+5}{z+3}}$$

$$=\frac{z+2}{z+3}\times\frac{z+3}{2z+5}$$

$$T_2T_1(z)=\frac{z+2}{2z+5}$$

iv) $T_1^{-1}T_2(z)=T_1^{-1}\left(\frac{z}{z+1}\right)$

$$=\frac{2-3\left(\frac{z}{z+1}\right)}{\left(\frac{z}{z+1}\right)-1}$$

$$=\frac{2-\frac{3z}{z+1}}{\frac{z-z-1}{z+1}}$$

$$=\frac{2z+2-3z}{z+1}\times\frac{z+1}{-1}$$

$$=\frac{2-z}{-1}$$

$$T_1^{-1}T_2(z)=z-2.$$

v) $T_1^{-1}T_1(z)=T_1^{-1}\left(\frac{z+2}{z+3}\right)$

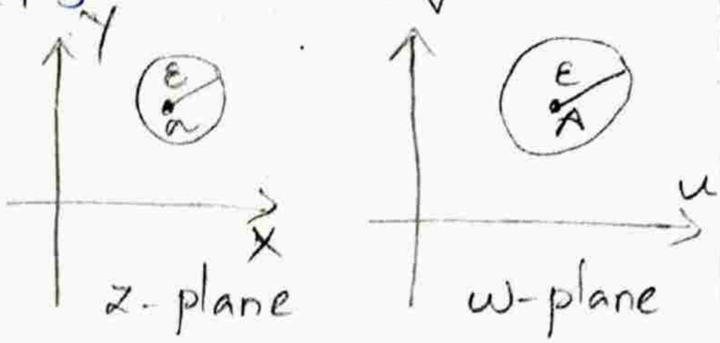
$$=\frac{2-3\left(\frac{z+2}{z+3}\right)}{\frac{z+2}{z+3}-1}$$

$$= \frac{2 - \left(\frac{3z+6}{z+3}\right)}{\frac{z+2-z-3}{z+3}}$$

$$= \frac{2z+6-3z-6}{z+3} \times \frac{z+3}{-1}$$

$$= \frac{-z}{-1}$$

$$T_1^{-1} T_1(z) = z.$$



Limits and continuity

The function $f(x)$ is said to have the limit A. As x tends to a.
$\lim_{x \to a} f(x) = A$ iff the following is true for every $\varepsilon > 0$ there exists a number $\delta > 0$ with the property that $|f(x) - A| < \varepsilon$ for all values of x such that $|x - a| < \delta$ and $x \neq a$.

Example:

$$\lim_{x \to 2} \frac{z^2 - 4}{z - 2} = 4$$

Soln:

$$f(z) = \frac{z^2 - 4}{z - 2}$$

$$= \frac{(z+2)(z-2)}{z-2} = z+2$$

$$\lim_{x \to 2} f(z) = \lim_{x \to 2} z+2$$
$$= 2+2$$
$$= 4$$

properties of limits

i) $\lim_{x \to a} \overline{f(x)} = \bar{A}$

ii) $\lim_{x \to a} Re\, f(x) = Re\, A$

iii) $\lim_{x \to a} Im\, f(x) = Im\, A$.

continuous:

The function $f(x)$ is said to be continuous at 'a' iff $\lim_{x \to a} f(x) = f(a)$.
Thus $f(x)$ is continuous at $f(a)$ if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Ex:
$f(z) = xy^2 + i(2x-y)$ is continuous type every where in the complex field.

properties of continuous

Let $f(x)$, $g(x)$ is continuous

i) $f(x) + g(x)$ is continuous.

ii) $f(x)g(x)$ is continuous

iii) $\frac{f(x)}{g(x)}$ is continuous in $g(x) \neq 0$

iv) $Re\, f(x)$, $Im\, f(x)$, $|f(x)|$ are continuous.

Differentiable:

Let f be a complex function define in a region D and let $z \in D$. Then f is said to be "differentiable of z". If $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists and its finite.

This limit is denoted by $f'(z)$ and this is called the derivative of $f(z)$ at z.

The function is said to be differentiable D if it is differentiable at all points of D

Ex: The function $f(z) = z^2$ is differentiable everywhere

Analytic function (or) holonomic function

A function 'f' define in a region D of the complex plane is said to be analytic at a point $a \in D$ if 'f' is differentiable at every point of some neighbourhood of a.

Thus f is analytic at A if there exists $\varepsilon > 0$ such that f is differentiable at every point of the disc

$$\delta(a, \varepsilon) = \{ z / |z - a| < \varepsilon \}$$

If f is analytic at every point of the region D. Then f is said to be analytic in D.

The function which is analytic at every point of complex plane is called an entire function or integral function.

Ex: Any polynomial is an entire function $z^3 + z$ is analytic

Cauchy Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{(or)} \quad U_x = V_y$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad\quad U_y = -V_x.$$

Laplace equation

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Harmonic function

A function u is satisfies Laplace eqn

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ is said to be harmonic function.

Ex: $u = x^2 - y^2$

soln:-

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

Hence $u = x^2 - y^2$ is harmonic.

conjugate harmonic function

If two harmonic function u and v satisfies CR-equations $U_x = V_y$ and $U_y = -V_x$. Then v is said to be harmonic conjugate of u.

pblm:

verify CR-eqns for the function

i) z^2 ii) e^z iii) z^3 iv) $\text{Re}\, z$ v) $e^x(\cos y - i\sin y)$

soln:

i) Let $z = x + iy$

$$w = f(z) = z^2 = (x+iy)^2$$
$$= x^2 - y^2 + 2ixy$$

Equating Re and Im parts

$$u = x^2 - y^2 \qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$f(z) = z^2$ satisfies CR-equations.

ii) $w = f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$

$$= e^x (\cos y + i \sin y)$$

Equating real and Im parts

$u = e^x \cos y \qquad v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$w = e^z = e^x(\cos y + i \sin y)$ satisfies the CR equations.

iii) $w = f(z) = z^3 = (x+iy)^3$

$$= x^3 - iy^3 + 3x^2 iy - 3xy^2$$

Equating real and Im parts

$u = x^3 - 3xy^2 \qquad v = 3x^2 y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore f(z) = z^3$ satisfies $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

iv) $w = \text{Re } z$

Let $z = x + iy$

Now, $u = x \qquad v = 0$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 0$

Re z not satisfy the CR eqns.

v) $w = f(z) = e^{x}(\cos y - i\sin y)$

By parts of Re and Im

$$u = e^{x}\cos y \qquad v = -e^{x}\sin y$$

$$\frac{\partial u}{\partial x} = e^{x}\cos y \qquad \frac{\partial v}{\partial x} = -e^{x}\sin y$$

$$\frac{\partial u}{\partial y} = -e^{x}\sin y \qquad \frac{\partial v}{\partial y} = -e^{x}\cos y$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

Hence $f(z) = e^{x}(\cos y - i\sin y)$ does not satisfies CR eqns.

P.T a harmonic function u satisfies the formal differential eqn $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Soln:-

Givn u is harmonic

ie) u satisfy Laplace eqn

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Now, $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial x}{\partial z} = 1/2 \qquad \frac{\partial y}{\partial z} = 1/2i$$

$$\frac{\partial x}{\partial \bar{z}} = 1/2 \qquad \frac{\partial y}{\partial \bar{z}} = -1/2i$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2}\frac{\partial u}{\partial x} + \left(-1/2i\right)\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \bar{z}} = 1/2\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= 1/2\frac{\partial u}{\partial x} + (1/2i)\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial z} = 1/2\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar{z}}\right) = \frac{\partial}{\partial z}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{\partial^2 u}{\partial z \partial x} + i\frac{\partial^2 u}{\partial z \partial y}\right)$$

$$= \frac{1}{2}\left\{\frac{\partial}{\partial x}\frac{\partial x}{\partial z}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}\left(\frac{\partial u}{\partial x}\right)\right.$$

$$\left. + i\left[\frac{\partial}{\partial x}\frac{\partial x}{\partial z}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right]\right\}$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\cdot\frac{1}{2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{2i}\right)\left(\frac{\partial u}{\partial x}\right) + \right.$$

$$\left. i\frac{\partial}{\partial x}\frac{1}{2}\left(\frac{\partial u}{\partial y}\right) + i\frac{\partial}{\partial y}\left(\frac{1}{2i}\right)\left(\frac{\partial u}{\partial y}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial x \partial y} + i\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

Hence proved.

Necessary and sufficient condition for the function $f(z)$ to be analytic.

Necessary condition:

Let $w = f(z) = u(x,y) + iv(x,y)$ be differentiable at any point $z = x+iy$ of its domain D. Then the partial derivatives u_x, u_y, v_x, v_y exists and satisfy the CR eqn $u_x = v_y$ and $u_y = -v_x$

proof:- Let $f(z) = u+iv$ be analytic at any point z of the domain D.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists}$$

and is unique.

If $z = x+iy$, $\Delta z = \Delta x + i\Delta y$ as $\Delta z \to 0$

Δx and Δy also tends to zero

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + \frac{iv(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y} \right] \quad \text{--- (1)}$$

Now, let us take possible path in which $\Delta z \to 0$.

case (i):

$\Delta z \to 0$ along the real axis such that $\Delta z = \Delta x$

$\Delta y = 0$ and allow $\Delta x \to 0$.

we get

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{iv(x+\Delta x) - v(x,y)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right] + i \lim_{\Delta x \to 0} \left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$f'(z) = u_x + i v_x \quad \text{--- (2)}$$

Since $f'(z)$ exists the above limit exists which means that u_x and v_x exists.

case (ii):

$\Delta z \to 0$ along the imaginary axis such that $\Delta z = i\Delta y$, $\Delta x = 0$ and allow $\Delta y \to 0$

from (1) we get

$$f'(z) = \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + i\,\frac{v(x, y+\Delta y) - v(x, y)}{i\Delta y} \right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} \right] + \lim_{\Delta y \to 0} \left[\frac{v(x, y+\Delta y) - v(x, y)}{i\Delta y} \right]$$

$$= \frac{1}{i} u_y + v_y$$

$$= -iu_y + v_y$$

$$f'(z) = v_y - iu_y \quad \text{— (3)}$$

Since $f'(z)$ exists the above limit exists which means u_y and v_y exists.

Since the limit should be unique and hence eqn (2) and (3) should be equal.

$$u_x + iv_x = v_y - iu_y.$$

Equating real and Im parts

$$u_x = v_y \text{ and } u_y = -v_x$$

It's eqn are called CR-eqns.

Sufficient condition:

The function $w = f(z) = u + iv$ is analytic is a domain D. If (i) u, v are differentiable in D and $u_x = v_y$ and $u_y = -v_x$.

(ii) The partial derivative u_x, u_y, v_x, v_y are all continuous in D.

proof:

Let $w = u + iv$ — (1)

$\Delta w = \Delta u + i\Delta v$ — (*)

Now, $\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\} \quad \text{—} ②$$

using mean value thm in ② we get

$$\Delta u = \Delta x\, u_x\, (x+\theta_1 \Delta x, y+\Delta y) + \Delta y\, u_y\, (x, y+\theta_2 \Delta y) \quad \text{—} ③$$

where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$

Since u_x is gvn to be continuous

$|u_x(x+\theta, \Delta x, y+\Delta y) - u_x(x,y)|$

III^{ly}, since u_y is continuous

$|u_y(x, y+\theta_2 \Delta y) - u_y(x,y)| < \eta$

Let us choose $\varepsilon_1 < \varepsilon$ and $\eta_1 < \eta$

$$u_x(x+\theta_1 \Delta x, y+\Delta y) - u_x(x,y) = \varepsilon_1$$

$$u_x(x+\theta_1 \Delta x, y+\Delta y) = \varepsilon_1 + u_x(x,y)$$

III^{ly} $u_y(x, y+\theta_2 \Delta y) = \eta_1 + u_y(x,y)$

sub in ③

$$\Delta u = \Delta x\,(u_x(x,y) + \varepsilon_1) + \Delta y\,(u_y(x,y) + \eta_1) \quad \text{—} ④$$

In the same way, we get,

$$\Delta v = \Delta x\,(u_x(x,y) + \varepsilon_2) + \Delta y\,(u_y(x,y) + \eta_2) \quad \text{—} ⑤$$

putting these values in (*)

$$\Delta w = [(u_x + \varepsilon_1)\Delta x + (u_y + \eta_1)\Delta y] + i\,[(v_x + \varepsilon_2)\Delta x + (v_y + \eta_2)\Delta y]$$

$$= [u_x \Delta x + u_y \Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y + i v_x \Delta x + \varepsilon_2 \Delta x + i \Delta y\, v_y + i \eta_2 \Delta y]$$

$$= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + (\varepsilon_1 + i\varepsilon_2)\Delta x + (\eta_1 + i\eta_2)\Delta y$$

$$= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + \varepsilon_0 \Delta x + \eta_0 \Delta y$$

using $u_x = v_y$ and $u_y = -v_x$ and

$\varepsilon_0 = \varepsilon_1 + i\varepsilon_2$ and $\eta_0 = \eta_1 + i\eta_2$

$$\Delta w = (u_x + iv_x)\Delta x + (-v_x + iu_x)\Delta y + \varepsilon_0 \Delta x + \eta_0 \Delta y$$

$$= (u_x + iv_x)\Delta x + (i^2 v_x + iu_x)\Delta y + \varepsilon_0 \Delta x + \eta_0 \Delta y$$

$$= (u_x + iv_x)\Delta x + i(u_x + iv_x)\Delta y + \varepsilon_0 \Delta x + \eta_0 \Delta y$$

$$= (u_x + iv_x)(\Delta x + i\Delta y) + \varepsilon_0 \Delta x + \eta_0 \Delta y$$

Divide by $\Delta z = \Delta x + i\Delta y$

$$\frac{\Delta w}{\Delta z} = (u_x + iv_x) + \frac{\varepsilon_0 \Delta x + \eta_0 \Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = (u_x + iv_x) + \lim_{\Delta z \to 0} \left(\frac{\varepsilon_0 \Delta x + \eta_0 \Delta y}{\Delta x + i\Delta y} \right)$$

as $\Delta z \to 0$, $\Delta x \to 0$, $\Delta y \to 0$, $\varepsilon_0 \to 0$, $\eta_0 \to 0$

$$\frac{dw}{dz} = u_x + iv_x$$

ie) $f'(z) = u_x + iv_x$

Since u_x, v_x exists and are unique.

$f'(z)$ exists

Hence $f(z)$ is analytic at an arbitrary point z.

$\therefore f(z)$ is analytic in the region.

polynomial

A polynomial is one complex variable is an expression of the form $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ where a_i's are complex number for $1 \le i \le n$

Note:

The identity z is not constant analytic function in the whole plane in derivative Generally the function z^n is analytic in the whole plane with derivative nz^{n-1}.

Since the sums and products of analytic functions are again analytic. The polynomial $p(z)$ is analytic in the whole plane derivatives ~~$a_1+2a_2z+\cdots$~~

$a_1 + 2a_2z + \cdots + na_nz^{n-1}$.

Rational function

A rational function is a function of the form $R(z) = \frac{P(z)}{Q(z)}$, where $p(z)$ and $Q(z)$ are two polynomials having no common factors.

Note:-

The order of a rational function is the common number of zeros and poles.

The rational function of order 1 is called the linear fractional transformation.

pblm: 1

If Q is a polynomial of degree n with distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and if P is a polynomial of degree $< n$. Show that

$$\frac{p(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}$$

Soln:- Let $Q(z) = c(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$ with $c \neq 0$

$$\frac{p(z)}{Q(z)} = \frac{P(z)}{c(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}$$

$$c \cdot \frac{P(z)}{Q(z)} = \sum_{i=1}^{n} \frac{A_i}{(z-\alpha_i)} \quad \text{—— ①}$$

when the right side represents the partial fraction expansion of $c\,\frac{P(z)}{Q(z)}$

$$c\,\frac{P(z)}{Q(z)} = \frac{\sum_{j=1}^{n} A_j \cdot \prod_{i \neq j} (z-\alpha_i)}{Q(z)/c}$$

$$p(z) = \sum_{j=1}^{n} A_j \prod_{i \neq j} (z-\alpha_i)$$

Hence $p(\alpha_K) = A_K \prod_{i \neq K} (\alpha_K - \alpha_i)$

$$A_K = \frac{p(\alpha_K)}{\prod_{i \neq K} (\alpha_K - \alpha_i)} \quad \text{—— ②}$$

Now, $Q'(z) = c \prod_{i \neq j} (z-\alpha_i)$

$$Q'(\alpha_K) = c \prod_{i \neq K} (\alpha_K - \alpha_i) \quad \text{—— ③}$$

sub ③ in ②

$$A_K = c\,\frac{P(\alpha_K)}{Q'(\alpha_K)}$$

From ①, $$c\,\frac{P(z)}{Q(z)} = \sum_{K=1}^{n} c\,\frac{P(\alpha_K)}{Q'(\alpha_K)(z-\alpha_K)}$$

$$\frac{p(z)}{Q(z)} = \sum_{K=1}^{n} \frac{p(\alpha_K)}{Q'(\alpha_K)(z-\alpha_K)}$$

Hence proved.

Convergence sequence

The sequence $\{a_n\}^{\infty}$ has the limit A if to every $\varepsilon > 0$ there exist an n_0 ~~number~~ such that $|a_n - A| < \varepsilon$ for $n \geq n_0$.

A sequence with a finite limit is said to be convergence, an any sequence which does not convergent is divergent.

If $\lim_{n \to \infty} a_n = \infty$ the sequence is said to be diverge to infinity.

cauchy sequence or fundamental sequence

A sequence will be called fundamental or a cauchy sequence if it satisfies the following condition.

Given any $\varepsilon > 0$ there exists an n_0 such that $|a_n - a_m| < \varepsilon$, whenever $n \geq n_0$ and $m, n \geq n_0$.

Thm:

A sequence is convergent iff it is a cauchy sequence.

proof:-

Let $\{a_n\}$ be a convergent sequence and converging to A.

ie) $\{a_n\} \to A$.

For given $\varepsilon > 0$ there exists n_0 such that $|a_n - A| < \varepsilon/2 \quad \forall \ n \geq n_0$.

To prove, $\{a_n\}$ is a cauchy sequence

Let $n, m \geq n_0$.

Now, consider
$$|a_n - a_m| = |a_n - A + A - a_m|$$
$$\leq |a_n - A| + |A - a_m|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

$\therefore |a_n - a_m| < \varepsilon \quad \forall \ n, m \geq n_0$.

The sequence $\{a_n\}$ is a cauchy sequence.

conversely,

let the sequence $\{x_n\}$ be a real cauchy sequence.

For given $\varepsilon > 0$ there exist n_0 such that $|x_n - x_m| < \varepsilon$

$\Rightarrow |x_n| < |x_m| + \varepsilon \quad \forall \quad n \geq n_0$

Now, $A = \lim_{n \to \infty} \sup x_n$.

$$a = \lim_{n\to\infty} \inf x_n.$$

Let it be both finite.

Suppose $A \neq a$.

choose $2\varepsilon = A - a$

By defn, we have $x_n < a + \varepsilon$ and $x_m < A - \varepsilon$.

Now consider,

$$|A - a| = |A - x_m + x_m - x_n + x_n - a|$$
$$\leq |A - x_m| + |x_m - x_n| + |x_n - a|$$
$$< \varepsilon + \varepsilon + \varepsilon$$
$$< 3\varepsilon$$

$|A - a| < 3\varepsilon$ which is contradiction.

$\therefore A = a$

ie) $\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \inf x_n$

$\therefore \{x_n\}$ is a convergent sequence.

Hence proved.

Necessary and sufficient condition for Σa_n to be convergent.

The series $\sum_{n=1}^{\infty} a_n$ converges iff for every $\varepsilon > 0$ there exist n_0 such that $|a_n + a_{n+1} + \cdots + a_{n+p}| < \varepsilon \ \forall\ n \geq n_0, p \geq 0$.

Absolute convergent

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Uniform convergent

The sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ on the set E is to every $\varepsilon > 0$ there exist an n_0 such that $|f_n(x) - f(x)| < \varepsilon \ \forall \ n \geq n_0$ and all x in E.

Thm: The limit function of a uniformly convergent sequence of continuous function of itself continuous.

proof:- Suppose that the function $f_n(x)$ are continuous and uniformly converges to 'a' of set E.

For any $\varepsilon > 0$ there exist an n_0 such that $|f_n(x) - f(x)| < \varepsilon/3 \ \forall \ x \in E$ — ①

Let x_0 be a point on E.

Since $f_n(x)$ is a continuous at x_0 there exist $\delta > 0$ such that

$|f_n(x) - f_n(x_0)| < \varepsilon/2$ — ② $\forall \ x \in E$

with $|x - x_0| < \delta$

Now,

$$
\begin{aligned}
|f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\
&\leq |f_n(x) - f(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&< \varepsilon
\end{aligned}
$$

$\therefore |f(x) - f(x_0)| < \varepsilon$

Hence $f(x)$ is continuous at x_0.

Cauchy necessary and sufficient condition for the function to be uniform converges

The sequence $\{f_n(x)\}$ converges uniformly on $E \iff$ for every $\varepsilon > 0$ there exist an n_0 such that $|f_m(x) - f_n(x)| < \varepsilon$ $\forall$ $m, n \geq n_0$ and all x in E.

Thm :

An analytic function in a region Ω whose derivative vanishes identically must reduces to a constant. This says prove if either real part imaginary part, the modulus and the argument is constant function.

proof:- part - I

Let $f(z) = u + iv$

The vanishing of a derivatives are

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} = 0$$

u, v are constant in any line segment in Ω which is parallel to one of its co-ordinate axis.

W.K.T, two points in a region can be joined within the region by a polygon whose sides are parallel to a axis

$\therefore$ $u + iv$ is a constant

$\therefore$ $f(z)$ is constant.

part - II

The imaginary part (or) the real part are constant.

Suppose the real part u is constant.

$$\frac{\partial u}{\partial x} = 0 \; ; \; \frac{\partial u}{\partial y} = 0$$

But f is analytic

$\therefore$ u, v satisfy CR-equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$\Rightarrow \frac{\partial v}{\partial y} = 0$ & $\frac{\partial v}{\partial x} = 0$

$\Rightarrow$ v is constant

$\Rightarrow$ u+iv is constant

$\therefore$ f(z) is constant.

If the imaginary part v is constant then as in the above procedure u is constant.

u+iv is constant

$\therefore$ f(z) is constant.

part - III

Suppose the modulus is constant then $|f(z)| = \sqrt{u^2+v^2}$ is constant also u^2+v^2 is constant.

case i) If $u^2+v^2 = 0$

$\Rightarrow u^2 = 0$ & $v^2 = 0 \Rightarrow u = 0$ & $v = 0$

$\Rightarrow f(z) = u+iv \Rightarrow 0+i0 = 0$

ie) $f(z) = 0$ is constant

f(z) is constant.

case ii) If $u^2+v^2 \neq 0$

$\Rightarrow u^2+v^2 = k \quad (k \neq 0) \rightarrow (*)$

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u \cdot \frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$$

$$u \cdot \frac{\partial u}{\partial x} + v\left(-\frac{\partial v}{\partial y}\right) = 0 \longrightarrow (1)$$

Diff (*) w.r. to 'y'

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \Rightarrow u\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\cdot v = 0 \rightarrow (2)$$

Solve (1) & (2)

(1)×u

$$① \times v \Rightarrow \quad uv \cdot \frac{\partial u}{\partial x} - v^2 \frac{\partial u}{\partial y} = 0$$

$$uv \frac{\partial u}{\partial x} + u^2 \frac{\partial u}{\partial y} = 0$$

$$-u^2 \frac{\partial u}{\partial y} - v^2 \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial y}(-u^2 - v^2) = 0$$

$$\frac{\partial u}{\partial y} = 0$$

$\int$ing $\quad u = $ constant

$|||^{ly}$ $\quad \frac{\partial v}{\partial y} = 0 \Rightarrow v$ is constant

$u + iv$ is constant

$\therefore f(z)$ is contant.

part - iv

$\arg f(z) = \tan^{-1}\left(\frac{v}{u}\right) = $ constant

$\Rightarrow \frac{v}{u} = $ constant

$\Rightarrow \frac{v}{u} = k \quad$ (k is constant)

$\Rightarrow v = uk$

$v_x = k u_x \quad \& \quad v_y = k u_y$

Eliminating k from the above eqns

$$\frac{v_x}{u_x} = \frac{v_y}{u_y} \Rightarrow v_x u_y = u_x v_y$$

$$\Rightarrow u_y(u_y) = u_x(-u_x)$$

$$u_y^2 + u_x^2 = 0$$

$$\Rightarrow u_x = 0 \text{ and } u_y = 0$$

$\therefore u$ is constant.

$|||^{ly}$ $\quad v$ is constant $\Rightarrow u + iv$ is constant

$\therefore f(z)$ is constant.

Thm:

Prove that the convergent sequence is bounded.

proof:-

Let $\{p_n\}$ is convergence sequence is convergent to P

ie) $P_n \to P$

there is an integer $N \ni: n > N$.

$$d(P_n, P) < 1$$

Choose $r = \max\{1, d(P_1, P), d(P_2, P), \dots, d(P_N, P)\}$

then,

$$d(P_n, P) \le r \quad \text{for } n = 1, 2, \dots$$

Hence $\{P_n\}$ is bounded

Thm:

The linear transformation S which maps 3 distinct points z_2, z_3, z_4 into $1, 0, \infty$ this order is unique.

proof:

Given S is a linear transformation which carries z_2, z_3, z_4 into $1, 0, \infty$

$$S(z_2) = 1, \quad S(z_3) = 0, \quad S(z_4) = \infty$$

Suppose T be a linear transformation which also maps the points z_2, z_3, z_4 into $1, 0, \infty$ in this order

$$T(z_2) = 1 \quad T(z_3) = 0 \quad T(z_4) = \infty$$

consider a transformation ST^{-1}

$$\left.\begin{aligned} ST^{-1}(1) &= ST^{-1}T(z_2) = S(z_2) = 1 \\ ST^{-1}(0) &= ST^{-1}T(z_3) = S(z_3) = 0 \\ ST^{-1}(\infty) &= ST^{-1}T(z_4) = S(z_4) = \infty \end{aligned}\right\} \to (1)$$

Let $ST^{-1}(z) = \dfrac{az+b}{cz+d} \to (2)$

$$ST^{-1}(1) = \frac{a+b}{c+d}$$

$$c+d = a+b \longrightarrow (3)$$

$$ST^{-1}(0) = b/d \Rightarrow 0 = b/d$$

$$\Rightarrow b = 0 \longrightarrow (4)$$

$$ST^{-1}(z) = \frac{z(a+b/z)}{z(c+d/z)} = \frac{a+b/z}{c+d/z}$$

$$ST^{-1}(\infty) = a/c \Rightarrow \infty = a/c \Rightarrow c=0 \rightarrow (5)$$

sub (4) and (5) in (3)

$$d=a \Rightarrow ST^{-1}(z) = \frac{az+b}{cz+d}$$

$$= \frac{dz+0}{0+d}$$

$$ST^{-1}(z) = \frac{dz}{d} = z$$

$$S(z) = T(z)$$

$$S = T$$

Hence S is a unique transformation which carries z_2, z_3, z_4 into $1, 0, \infty$ in this order.

Thm:

If z_1, z_2, z_3, z_4 are distinct points in the extended plane on T be any linear transformation then

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$$

proof:- ~~Let~~ consider $S(z_1) = (z_1, z_2, z_3, z_4)$

$$S(z_2) = 1, \quad S(z_3) = 0, \quad S(z_4) = \infty$$

Let T be a linear transformation which maps z_2, z_3, z_4 into $1, 0, \infty$ in order

$$T(z_2) = 1, \quad T(z_3) = 0, \quad T(z_4) = \infty$$

consider, a linear transformation, ST^{-1}

$$ST^{-1}T(z_2) = S(z_2) = 1$$

$$ST^{-1}T(z_3) = S(z_3) = 0$$

$$ST^{-1}T(z_4) = S(z_4) = \infty$$

ST^{-1} is a linear transformation which maps Tz_2, Tz_3, Tz_4 into $1, 0, \infty$

$$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}T(z_1) = S(z_1)$$

$$= (z_1, z_2, z_3 z_4)$$

$$\therefore (Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$$

Hence proved.

Thm: A linear transformation carries circle into circles.

proof:-

Let z_1, z_2, z_3, z_4 be the points on the circle c.

We know that, let z_1 be any arbitrary point on c.

"The cross ratio $(z_1\ z_2\ z_3\ z_4)$ is real iff the four points lie on the circle or on a straight line" $\rightarrow$ ①

By (1) $(z_1\ z_2\ z_3\ z_4)$ is real.

We know that, "previous thm" $\rightarrow$ ②

By ①

$(z_1\ z_2\ z_3\ z_4)$ is real.

By ②

$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1\ z_2\ z_3\ z_4)$

The R.H.S value is real

$\therefore (Tz_1, Tz_2, Tz_3, Tz_4)$ is also real.

The image of z_1, z_2, z_3, z_4 are lie on a common circle.

Thm The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on the circle (or) concyclic (or) on a straight line.

proof:- This is evident by elementary geometry for we obtain,

$$\arg(z_1\ z_2\ z_3\ z_4) = \arg\left(\frac{z_1 - z_3}{z_1 - z_4}\right) - \arg\left(\frac{z_2 - z_3}{z_2 - z_4}\right) \rightarrow (1)$$

and if the points lie on a circle, this difference of angle is either 0 (or) $\pm\pi$ depending on the relative location

$$(1)\ \arg(z_1\ z_2\ z_3\ z_4) = \arg\left(\frac{z_1 - z_3}{z_1 - z_4}\right) / \arg\left(\frac{z_2 - z_3}{z_2 - z_4}\right)$$

$\Rightarrow \arg(z_1\ z_2\ z_3\ z_4)$ is real

Now, we need only show that the image of the real axis under any linear transformation is either a circle (or) a straight line.

$Tz_1 = (z_1, z_2, z_3, z_4)$ is real on the image of the real axis under the transformation T^{-1}.

The values of $w = T^{-1}(z)$ for real z satisfy the eqn.

$$Tw = \overline{Tw}$$

$$\frac{aw+b}{cw+d} = \left(\overline{\frac{aw+b}{cw+d}}\right) = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$$

$$(aw+b)(\bar{c}\bar{w}+\bar{d}) = (\bar{a}\bar{w}+\bar{b})(cw+d)$$

$$\Rightarrow aw\bar{w}\bar{c} + b\bar{c}\bar{w} + a\bar{d}w + b\bar{d} = \bar{a}\bar{w}cw + \bar{a}\bar{w}d + \bar{b}cw + \bar{b}d$$

$$\Rightarrow a\bar{c}w\bar{w} + a\bar{d}w + b\bar{c}\bar{w} + b\bar{d} - \bar{a}cw\bar{w} - \bar{a}d\bar{w} - \bar{b}cw - \bar{b}d = 0$$

$$\Rightarrow (a\bar{c}-\bar{a}c)w\bar{w} + (a\bar{d}-\bar{b}c)w + (b\bar{c}-\bar{a}d)\bar{w} + (b\bar{d}-\bar{b}d) = 0 \rightarrow (2)$$

put $a\bar{c}-\bar{a}c = 0$ in (2)

$$(a\bar{d}-\bar{b}c)w + (b\bar{c}-\bar{a}d)\bar{w} + (b\bar{d}-\bar{b}d) = 0 \rightarrow (3)$$

$$\Rightarrow \bar{\alpha}w + \alpha\bar{w} + \beta = 0 \rightarrow (4)$$

where β is purely imaginary and $\bar{\alpha} = a\bar{d}-\bar{b}c$ and $\alpha = b\bar{c}-\bar{a}d$ and $\beta = b\bar{d}-\bar{b}d$

Equating (4) is the standard form of straight line

Take, $a\bar{c}-\bar{a}c \neq 0$

$\div\ a\bar{c}-\bar{a}c$

$$(2) \Rightarrow w\bar{w} + \left[\frac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}c}\right]w + \left[\frac{b\bar{c}-\bar{a}d}{a\bar{c}-\bar{a}c}\right]\bar{w} + \left[\frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}\right] = 0 \rightarrow (5)$$

$$\Rightarrow w\bar{w} + \bar{b}w + b\bar{w} + c = 0$$

which is the general eqn of the circle with centre $-b$ and radius $\sqrt{b\bar{b}-c}$.

$$\bar{b} = \frac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}c}\ ;\ b = \frac{b\bar{c}-\bar{a}d}{a\bar{c}-\bar{a}c}\ ;\ c = \frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}$$

$$\text{centre } (-b) = -\left(\frac{b\bar{c}-\bar{a}d}{a\bar{c}-\bar{a}c}\right) = \frac{\bar{a}d-b\bar{c}}{a\bar{c}-\bar{a}c}$$

$$\text{radius} = \sqrt{b\bar{b}-c}$$

$$= \sqrt{\left(\frac{b\bar{c}-\bar{a}d}{a\bar{c}-\bar{a}c}\right)\left(\frac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}c}\right) - \left(\frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}\right)}$$

$$= \sqrt{\frac{ab\bar{c}\bar{d}-b\bar{b}c\bar{c}-a\bar{a}d\bar{d}+\bar{a}\bar{b}dc}{a\bar{a}\bar{c}\bar{c}-a\bar{a}c\bar{c}-a\bar{a}c\bar{c}+\bar{a}\bar{a}cc} - \frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}}$$

$$= \sqrt{\frac{ab\bar{c}\bar{d}-b\bar{b}c\bar{c}-a\bar{a}d\bar{d}+\bar{a}\bar{b}cd-[ab\bar{c}\bar{d}-a\bar{b}\bar{c}d-\bar{a}bc\bar{d}+\bar{a}\bar{b}cd]}{(a\bar{c}-\bar{a}c)^2}}$$

$$= \sqrt{\frac{-b\bar{b}c\bar{c} - a\bar{a}d\bar{d} + a\bar{b}\bar{c}d + \bar{a}bc\bar{d}}{(a\bar{c}-\bar{a}c)^2}}$$

$$= \sqrt{\frac{\bar{b}\bar{c}(ad-bc)+\bar{a}\bar{d}(bc-ad)}{(a\bar{c}-\bar{a}c)^2}} = \frac{\sqrt{\bar{b}\bar{c}(ad-bc)-\bar{a}\bar{d}(bc-ad)}}{a\bar{c}-\bar{a}c}$$

$$= \frac{\sqrt{(b\bar{c}-\bar{a}\bar{d})(ad-bc)}}{(a\bar{c}-\bar{a}c)}$$

$$\text{Radius} = \frac{\sqrt{(ad-bc)\,\overline{(ad-bc)}}}{a\bar{c}-\bar{a}c} = \frac{\sqrt{(ad-bc)^2}}{a\bar{c}-\bar{a}c}$$

$$= \frac{ad-bc}{a\bar{c}-\bar{a}c}$$

Eqn (5) is a circle with centre $\frac{\bar{a}d-b\bar{c}}{a\bar{c}-\bar{a}c}$ and radius $\frac{ad-bc}{a\bar{c}-\bar{a}c}$

$$\left| w + \frac{\bar{a}d-b\bar{c}}{\bar{a}c-\bar{a}c} \right| = \left| \frac{ad-bc}{\bar{a}c-a\bar{c}} \right|$$

which is the eqn of the circle

Hence proved.

Conformality

Axis and closed curves:-

The equation of an arc γ in the plane is given in parametric form $x = x(t)$ and $y = y(t)$ where t runs through as interval $\alpha \le t \le \beta$ and $x(t)$ and $y(t)$ are continuous functions.

Simple curve (or) Jordon curve

An arc is simple (or) Jordan arc if $z(t_1) = z(t_2)$ only for $t_1 = t_2$ ie) the function γ is 1-1.

Closed curve:

An arc is a closed curve if the end points coincide that is $z(\alpha) = z(\beta)$

Find the linear fractional transformation which carries $|z| = 1$ and $|z - 1/4| = 1/4$ onto concentric circle what is the ratio of the radii in the image plane.

Let T map these circles onto circles with centre 'a' and radii R_1 and R_2.

Since 0 and ∞ are symmetric with respect to both the image circles their pre-images say α and β must be symmetric with respect to both these given circles

Thus $\bar{\alpha}\beta = 1$ and $(\bar{\alpha} - 1/4)(\beta - 1/4) = 1/16$

$$\Rightarrow \bar{\alpha}\beta - \tfrac{1}{4}\beta - \bar{\alpha}\tfrac{1}{4} + 1/16 - 1/16 = 0$$

$(\bar{\alpha}\beta = 1)$ $\Rightarrow \bar{\alpha}\beta - 1/4(\bar{\alpha} + \beta) = 0$

$$(\bar{\alpha} + \beta)(-1/4) + 1 = 0$$

$$-1/4(\bar{\alpha} + \beta) = -1$$

$$\Rightarrow \bar{\alpha} + \beta = 4 \longrightarrow (1)$$

$$\Rightarrow \bar{\alpha} - \beta = \sqrt{(\bar{\alpha} + \beta)^2 - 4\bar{\alpha}\beta}$$

$$= \sqrt{4^2 - 4(1)} = \sqrt{16 - 4} = \sqrt{12}$$

$$\therefore \bar{\alpha} - \beta = \sqrt{12} = \pm 2\sqrt{3} \longrightarrow (2)$$

From (1) & (2) $\quad \bar{\alpha} + \beta = 4$

$$\bar{\alpha} - \beta = \pm 2\sqrt{3}$$

$$2\bar{\alpha} = 4 \pm 2\sqrt{3}$$

$$\bar{\alpha} = 2+\sqrt{3}$$

$$\bar{\alpha}+\beta = 4$$

$$\bar{\alpha}-\beta = \pm 2\sqrt{3}$$

$$2\beta = 4 \pm 2\sqrt{3}$$

$$\beta = 2 \pm \sqrt{3}$$

Hence T takes $2+\sqrt{3}$ to a and $2-\sqrt{3}$ to ∞ (or) $2+\sqrt{3}$ to ∞ $2-\sqrt{3}$ to a

One such linear fractional transformation is

$$Tz = \frac{z-(2\pm\sqrt{3})}{z-(2\mp\sqrt{3})} + a$$

Choose 1 on the unit circle and $1/2$ on the circle centre $1/4$ and radius $1/4$

Then $T(1)$ is on the first concentric circle and $T(1/2)$ is on the other concentric circle

Hence the ratio of the radii of the image circles $\frac{R_1}{R_2}$ is given by

$$\frac{R_1}{R_2} = \left|\frac{T(1)-a}{T(1/2)-a}\right| = 2-\sqrt{3}$$

Taking $\alpha = 2\pm\sqrt{3}$ and $\beta = 2-\sqrt{3}$

The other assumption $\alpha = 2-\sqrt{3}$ and $\beta = 2+\sqrt{3}$ gives $R_1/R_2 = 2+\sqrt{3}$

Defn: symmetric

The point z and z^* are said to symmetric with respect to the circle through z_1, z_2, z_3, z_4 iff $(z^*, z_1, z_2, z_3) = \overline{(z\,z_1, z_2, z_3)}$

Unit - II

* Complex integration

* Cauchy integral formula

Unit - II

Line integrals

Defn:

Let c be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \le t \le b$. Let $f(z)$ be a continuous complex valued function define in a region containing the curve c.

The line integral of $f(z)$ over c as

$$\int_c f(z)\,dz = \int_a^b f(z(t)z'(t))\,dt$$

problem: 1

Evalute $\int_c f(z)\,dz$ where $f(z) = y - x - i3x^2$ and c is the line segment from $z=0$ to $z=1+i$

Soln:-

- The equation of the line segoment c joining $z=0$ to $z=1+i$ is given by $y=z$

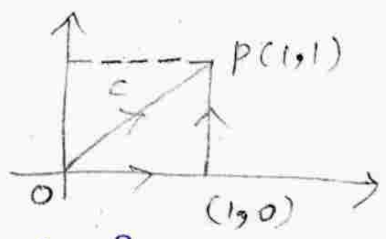
ie) The equation of op is $y=x$

$\therefore$ The parametric equation of c can be taken as $x=t$ and $y=t$ where $0 \le t \le 1$

$$z(t) = x(t) + iy(t)$$

$$z(t) = t + it$$

$$z'(t) = 1+i$$



Now, $f(z(t)) = t - t - i3t^2 = -i3t^2$

$$\int_c f(z)\,dz = \int_0^1 f(z(t))\,z'(t)\,dt$$

$$= \int_0^1 -i3t^2(1+i)\,dt$$

$$= -3i(1+i)\left[\frac{t^3}{3}\right]_0^1$$

$$= -3i(1+i)\left[\tfrac{1}{3}\right]$$

$= -3i(1+i)[1/3] = -i(1+i)$

$= -i-i^2 = -i+1 = 1-i$

$\int_c f(z)dz = 1-i$

problm: 2

Find the value of the line integral $\int_{|z|=1} |z-1|dz$

soln:-

$z = re^{i\theta} \Rightarrow r=1$

let $z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$|dz| = |ie^{i\theta}d\theta| \Rightarrow |i||e^{i\theta}|d\theta$

$= 1.1.d\theta \Rightarrow d\theta$

since $|i|=1$ and $|e^{i\theta}|=1$

Now, $z-1 = e^{i\theta}-1$

$= \cos\theta + i\sin\theta - 1$

$\Rightarrow \cos\theta - 1 + i\sin\theta$

$|z-1| = |\cos\theta - 1 + i\sin\theta|$

$= \sqrt{(\cos\theta-1)^2+\sin^2\theta}$

$= \sqrt{\cos^2\theta+1-2\cos\theta+\sin^2\theta}$

$= \sqrt{2-2\cos\theta}$

$= \sqrt{2(1-\cos\theta)}$

$= \sqrt{2\cdot 2\sin^2\theta/2}$

$= \sqrt{2^2\sin^2\theta/2}$

$|z-1| = 2\sin\theta/2$

$\int_{|z|=1} |z-1|dz = \int_0^{2\pi} 2\sin\theta/2 \, d\theta$

$= 2\left[\frac{-\cos\theta/2}{1/2}\right]_0^{2\pi}$

$|x+iy| = \sqrt{x^2+y^2}$

$i = 0+i$

$\Rightarrow |i| = \sqrt{0^2+1^2} = 1$

$e^{i\theta} = \cos\theta + i\sin\theta$

$|e^{i\theta}| = \sqrt{\cos^2\theta+\sin^2\theta} = \sqrt{1} = 1$

General parametric form of circle

$z = a + re^{i\theta}$

$|z| = 1 \Rightarrow |z-0| = 1$

$a=0;\ r=1$

$\Rightarrow z = 0 + e^{i\theta} - e^{i\theta}$

$= 2 \times 2 \left[-\left(\cos \frac{2\pi}{2} - \cos 0/2 \right) \right]$

$= 4 \left[-(\cos \pi - \cos 0) \right]$

$= 4 \left[-(-1) - 1 \right] \Rightarrow \int\limits_{|z|=1} |z-1| dz = 8$

Line integral as a function of arcs

The line integral is of the form $\int_\gamma p\,dx + q\,dy$ as a function of the arcs where p and q defined as continuous in the region Ω.

If γ_1 and γ_2 have the same initial points and end points then,

$$\int_{\gamma_1} p\,dx + q\,dy = \int_{\gamma_2} p\,dx + q\,dy.$$

Rectifiable arcs:

The length of the arc can also be defined as the least upper bound of all sums

$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \dots + |z(t_n) - z(t_{n-1})|$

where $a = t_0 < t_1 < t_2 < \dots < t_n = b$

If this least upper bound is finite we say that the arc is rectifiable.

V.Q Thm:

Necessary and sufficient condition under which the line integral depends only the end points.

Statement:

The line integral $\int_\gamma p\,dx + q\,dy$ defined in Ω depends only on the end points of γ iff there exists function $U(x,y)$ in

Ω with partial derivatives $\frac{\partial u}{\partial x} = p$; $\frac{\partial u}{\partial y} = q$.

proof:- Given that there exists a function $U(x,y)$ in Ω such that, $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$

To prove,

The line integral $\int_\gamma pdx + qdy$ depends only on the end points of γ.

$$\int_\gamma pdx + qdy = \int_\gamma \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= \int_a^b \left(\frac{\partial u}{\partial x} x'(t)dt + \frac{\partial u}{\partial y} y'(t)dt \right)$$

$$= \int_a^b \frac{d}{dt} U(x(t), y(t))dt$$

$$= \int_a^b d\, U\,(x(t), y(t))$$

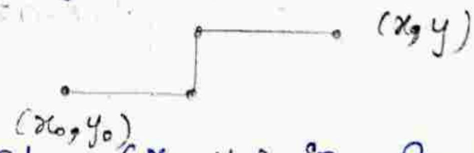
$$= U[x(t), y(t)]_a^b$$

$$\int_\gamma pdx + qdy = U[x(b), y(b)] - U[x(a), y(a)]$$

Hence the values of the integral depends on the end points.

conversely,

$\int_\gamma pdx + qdy$ depends on the end point of γ.



Let us fix a point (x_0, y_0) in Ω and join it to (x,y) by a polygon γ containing three line segment parallel to x-axis.

Let us define $U(x,y) = \int_\gamma pdx + qdy$

Since the line integral depends only on the end points of γ.

Clearly $u(x,y)$ is well defined in the last segment y is fixed $(dy=0)$ and x above various ie) $v(x,y) = \int^{x} p\,dx + q(0) + C$

where the lower limit is immaterial and c consist of this values of the integral from (x_0, y_0) to (x, y) along the polynomial path.

$$u(x,y) = \int^{x} p\,dx + c$$

$$\Rightarrow \frac{\partial u}{\partial x} = p$$

similarly by choosing the last segment parallel to y-axis

$$v(x,y) = \int^{y} q\,dy + c$$

$$\Rightarrow \frac{\partial u}{\partial y} = q.$$

Cauchy's theorem for a rectangle

If the function $f(z)$ is analytic on the rectangle R. Then $\int_{\partial R} f(z)\,dz = 0$

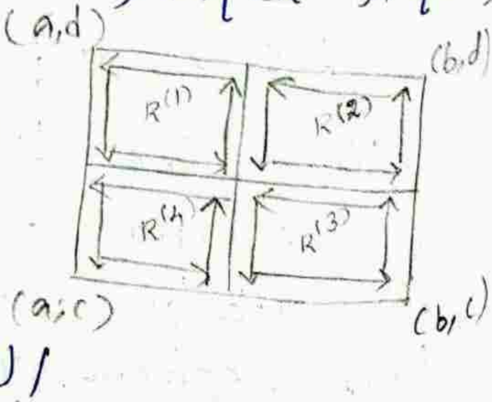
proof:-

Let us denote $\eta(R) = \int_{\partial R} f(z)\,dz$ divide the given rectangle into four complement rectangle $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$

ie) $$\int_{\partial R} f(z)\,dz = \int_{\partial R^{(1)}} f(z)\,dz + \int_{\partial R^{(2)}} f(z)\,dz + \int_{\partial R^{(3)}} f(z)\,dz + \int_{\partial R^{(4)}} f(z)\,dz$$

ie) $$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$$

For the integral over the common sides cancel, each other.



$$|\eta(R)| = |\eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})|$$

$\Rightarrow |\eta(R)| \le |\eta(R^{(1)})| + |\eta(R^{(2)})| + |\eta(R^{(3)})| + |\eta(R^{(4)})|$

There is atleast one rectangle R^k; $k = 1,2,3,4,\ldots$ must satisfies the condition

$$|\eta(R)| \le 4\,|\eta(R^k)|$$

$$|\eta(R^k)| \ge \frac{1}{4}\,|\eta(R)|$$

If there is more than one rectangle R^k choose some order to selection one rectangle as R_1

Then $|\eta(R_1)| \ge \frac{1}{4}|\eta(R)|$

Repeat the process, we have a sequence of rectangle $R \supset R_1 \supset R_2 \supset \ldots \supset R_n \supset \ldots$ with the property,

$$|\eta(R_n)| \ge \frac{1}{4}\,|\eta(R_{n-1})|$$

$$|\eta(R_n)| \ge \frac{1}{4}\cdot\frac{1}{4}\,|\eta(R_{n-2})|$$

$$|\eta(R_n)| \ge \frac{1}{4^2}\,|\eta(R_{n-2})|$$

$$\vdots \qquad \vdots$$

$$|\eta(R_n)| \ge \frac{1}{4^n}\,|\eta(R)| \longrightarrow (1)$$

The rectangle R_n converges to a point $z^* \in R$.

R_n is contained in the neighbourhood $|z - z^*| < \delta$ for n is sufficient large

We can choose δ sufficiently so small that,

i) $f(z)$ is defined and analytic in $|z - z^*| < \delta$

ii) Given $\varepsilon > 0$ we can choose δ so that,

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \varepsilon$$

$\Rightarrow |f(z) - f(z^*) - (z - z^*) f'(z^*)| < \varepsilon |z - z^*|$ ②

We can assume that δ satisfies both condition and that R_n is contained in $|z - z^*| < \delta$.

(We make now the observation that

$$\int_{\partial R_n} dz = 0, \quad \int_{\partial R_n} z\,dz = 0)$$

Now, $$|\eta(R_n)| = \int_{\partial R_n} |f(z) - f(z^*) - (z - z^*) f'(z^*)|\,|dz|$$

From ②

$$|\eta(R_n)| \leq \int_{\partial R_n} \varepsilon |z - z^*|\,|dz|$$

$z \in R_n$, $|z - z^*| < d_n$, where d_n is the diagonal of R_n

$$\int_{\partial R_n} |dz| = \text{perimeter of } R_n = L_n \text{ (say)}$$

$$\therefore |\eta(R_n)| \leq \varepsilon\, d_n L_n$$

d, L are diagonal and perimeter of the original rectangle R.

$$d_n = \frac{1}{2^n} d \quad \& \quad L_n = \frac{1}{2^n} L$$

$$|\eta(R_n)| \leq \varepsilon \frac{1}{4^n} dL$$

compare with ①

$$|\eta(R_n)| \geq \frac{1}{4^n} |\eta(R)|$$

$$|\eta(R)| \leq 4^n |\eta(R_n)|$$

$$\leq 4^n \varepsilon \frac{1}{4^n} \cdot dL = \varepsilon dL$$

$$|\eta(R)| \leq \varepsilon dL$$

Since ε is arbitrary $\eta(R) = 0$

Hence $\int_{\partial R} f(z)\,dz = 0$ proved

Thm: Let $f(z)$ be analytic on the set R' obtain from a rectangle R by omitting a finite number of interior points ζ_j if it is true that $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$ then $\int_{\partial R} f(z)\,dz = 0$.

proof: It is sufficient to consider a single expectational point ζ for evidently R can be divided into smaller rectangle which contains atmost one ζ_j

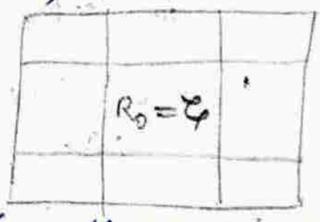
We divide R into a rectangles

Since f is analytic in R expect at ζ.

f is analytic in $R_1, R_2, \ldots$

By cauchy's rectangle theorem,

$$\int_{\partial R_i} f(z)\,dz = 0 \quad \forall\ i = 1, 2, \ldots, 8$$



$$\therefore \int_{\partial R} f(z)\,dz = \int_{\partial R_0} f(z)\,dz + \int_{\partial R_i} f(z)\,dz \qquad \text{Here } i = 1, 2, \ldots, 8$$

$$= \int_{\partial R_0} f(z)\,dz + 0$$

$$\therefore \int_{\partial R} f(z)\,dz = \int_{\partial R_0} f(z)\,dz \longrightarrow (1)$$

By hypothesis,

$$\lim_{z \to \zeta_j} (z - \zeta_j) f(z)\,dz = 0$$

If $\varepsilon > 0$ we can choose the rectangle R_0 so small that,

$$|f(z)| \le \frac{\varepsilon}{|z - \zeta|}$$

From (1)

$$\left| \int_{\partial R} f(z)\,dz \right| = \left| \int_{\partial R_0} f(z)\,dz \right|$$

$$\leq \int_{\partial R_0} |f(z)|\,|dz|$$

$$\leq \int_{\partial R_0} \frac{\varepsilon}{|z-c_1|}\,|dz|$$

We consider R_0 as a square with centre c_1 and side a_0

We have $|z-c_1| \geq a/2$

$$\Rightarrow \frac{1}{|z-c_1|} \leq 2/a$$

$$\therefore \left|\int_{\partial R} f(z)\,dz\right| \leq \frac{2\varepsilon}{a}\int_{\partial R_0}|dz| = \frac{2\varepsilon}{a} \times 4a = 8\varepsilon$$

$\because$ ε is arbitrary

$$\int_{\partial R} f(z)\,dz = 0 \,//$$

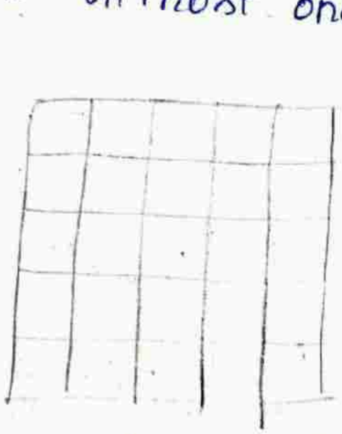
If the rectangle R has more than one exceptional point

ie) The finite number of exceptional points

We divide the rectangle R in such a way there

Each sub-rectangle contains atmost one exceptional points.

By cauchy's thm on a rectangle the line integrals over through subrectangle without exceptional points are zero.



By the 1^{st} case the line integral over the other subrectangle with exactly one exceptional points are also zero

$$\therefore \int_{\partial R} f(z)\,dz = 0$$

Hence proved.

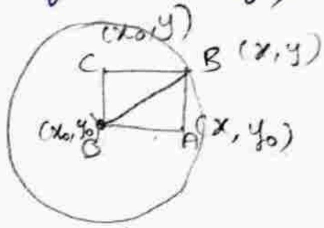
Cauchy's theorem in a disk

If $f(z)$ is analytic in an open disk (or circular disk Δ then $\int_\gamma f(z)dz = 0$ for every closed curve γ in Δ.

proof:- Let us consider a function,

$$F(z) = \int_\sigma f(z)dz$$

where σ consists of the horizontal line segment from (x_0, y_0) to (x, y_0) followed by the vertical line segment from (x, y_0) to (x, y)



$$F(z) = \int_\sigma f(z)dz$$

$$F(z) = \int_{OA} f(z)dz + \int_{AB} f(z)dz$$

$$= \int_{OA} f(z)(dx+idy) + \int_{AB} f(z)(dx+idy)$$

[Here O is the centre of the disk B be on the disk]

$$= \int_{OA} f(z)dx + i\int_{OA} f(z)dy + \int_{AB} f(z)dx + i\int_{AB} f(z)dy$$

[on OA y is fixed $dy=0$; on AB x is fixed $dx=0$].

$$\therefore F(z) = \int_{OA} f(z)dx + 0 + 0 + i\int_{AB} f(z)dy$$

$$F(z) = \phi(x) + i\int^y f(z)dy$$

Diff with respect to y, $\frac{\partial f}{\partial y} = if(z)$

$$\Rightarrow \frac{1}{i}\frac{\partial f}{\partial y} = f(z)$$

$$\Rightarrow -i\frac{\partial F}{\partial y} = f(z) \longrightarrow (1)$$

Let σ_1 denote the curve consisting of the vertical line segment from (x_0, y_0) to (x_0, y) followed by the horizontal line segment from (x_0, y) to (x, y)

$$F(z) = \int_{\sigma_1} f(z)dz$$

$$= \int_{OC} f(z)\,dz + \int_{CB} f(z)\,dz$$

$$= \int_{OC} f(z)(dx + i\,dy) + \int_{CB} f(z)(dx + i\,dy)$$

$$= \int_{OC} f(z)\,dx + i\int_{OC} f(z)\,dy + \int_{CB} f(z)\,dx + i\int_{CB} f(z)\,dy$$

[on OC x is fixed $dx = 0$; on CB y is fixed $dy = 0$]

$$F(z) = 0 + i\int_{OC} f(z)\,dy + \int_{CB} f(z)\,dx + 0$$

$$= i\int_{OC} f(z)\,dy + \int_{CB} f(z)\,dx$$

$$f(z) = i\,\phi(y) + \int^{x} f(z)\,dx$$

Diff w.r.to x, $\frac{\partial f}{\partial x} = f(z) \longrightarrow (2)$

From (1) & (2)

$$\therefore \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

which is the complex form of CR-eqns.

$\therefore$ f satisfies CR eqn.

Since f is analytic in Δ

$\Rightarrow$ f is differentiable at every point of Δ and hence it is a continuous function

$\frac{\partial F}{\partial x}$ & $\frac{\partial F}{\partial y}$ are continuous

$\therefore$ F is analytic in Δ

Thus $f(z)$ is the derivative of an analytic function $F(z)$

We know that,

"The integral $\int f(z)\,dz$ depends only on the end points of γ iff f is the derivative of an analytic function in Ω"

By the thm,

$\int_{\gamma} f(z)\,dz$ depends only on the end points of γ

We know that

" The line integral depends only on the end points iff the integral is exact differential"

$\therefore \int_\gamma f(z)dz$ is exact differential

$\therefore \int_\gamma f(z)dz = 0$, where γ is the closed ins

Thm:
The integral $\int_\gamma f(z)\,dz$ with continuous function f depends only on the end points of $\gamma \Leftrightarrow$ the f is derivative of an analytic function in Ω.

proof:
Suppose that line integral $\int_\gamma f(z)dz$ depends only on the end points of γ.

Let $z = x+iy$

$dz = dx+idy$

$\int_\gamma f(z)dz = \int_\gamma f(z)(dx+idy)$

$= \int_\gamma f(z)dx + i\int_\gamma f(z)dy$

This is a line integral depends only on the end points of γ then there exists a function $F(z)$ in Ω with the partial derivatives.

$$\frac{\partial F(z)}{\partial x} = f(z) \rightarrow (1)$$

$$\frac{\partial F(z)}{\partial y} = i f(z) = -i\frac{\partial f}{\partial y}(z) = f(z) \rightarrow (2)$$

From (1) & (2) $\Rightarrow \frac{\partial F(z)}{\partial x} = -i\frac{\partial F(z)}{\partial y}$

$$\Rightarrow \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

F(z) satisfies CR eqn.

Since f(z) is continuous

$\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are continuous

F is an analytic function

f is derivative of an ~~ab~~ analytic function F in Ω

conversely is similar.

Thm:

Let f(z) be analytic in the region Δ obtained by omitting a finite number of points ζ_j from an open disk Δ if f(z) satisfies the condition $\lim_{z \to \zeta_j} (z-\zeta_j) f(z) = 0$ $\forall j$ then $\int_\gamma f(z)dz$ holds for any closed curve γ in Δ.

proof:-

Let us consider the exceptional points ζ which does not lie on the line $x = x_0$

Let $A(x_0, y_0)$ be a centre of disk and $E(x,y)$ be any point on the disk.

Let us consider $F(z) = \int_{ABCDE} f(z)\,dz$

Here the line segment AB & DE are the vertical line segment and BD is the horizontal line segment and F(z) is independent of the middle segment.

Since the last line segment is vertical

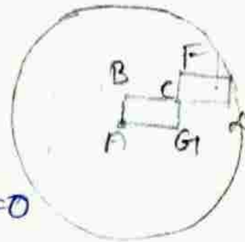
$$\frac{\partial F}{\partial y} = i f(z) \Rightarrow -i \frac{\partial F}{\partial y} = f(z) \rightarrow (1)$$

Now, join A and E be the polygon AGCFE as shown in figure.

This polygon are constructed such as

manner that the point ξ is an interior point of one of the rectangle.

By thm, we have



$$\int_{AGCBA} f(z)\,dz = 0 \quad \& \quad \int_{CDEFC} f(z)\,dz = 0$$

Now, $$\int_{AGCBA} f(z)\,dz = 0$$

$$\Rightarrow \int_{AGC} f(z)\,dz + \int_{CBA} f(z)\,dz = 0$$

$$\Rightarrow \int_{AGC} f(z)\,dz - \int_{ABC} f(z)\,dz = 0$$

$$\Rightarrow \int_{AGC} f(z)\,dz = \int_{ABC} f(z)\,dz$$

$\|^{ly}$ $$\int_{CDEFC} f(z)\,dz = 0$$

$$\Rightarrow \int_{CDE} f(z)\,dz + \int_{EFC} f(z)\,dz = 0$$

$$\Rightarrow \int_{CDE} f(z)\,dz - \int_{CFE} f(z)\,dz = 0$$

$$\Rightarrow \int_{CDE} f(z)\,dz = \int_{CFE} f(z)\,dz$$

$$f(z) = \int_{ABCDE} f(z)\,dz = \int_{ABC} f(z)\,dz + \int_{CDE} f(z)\,dz$$

$$= \int_{AGC} f(z)\,dz + \int_{CFE} f(z)\,dz$$

$$= \int_{AGCFE} f(z)\,dz$$

Since the last segment FE is horizontal

$$\frac{\partial f}{\partial x} = f(z) \rightarrow ②$$

From ① & ② $\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$

$\therefore f$ satisfies CR-eqn

F(z) is analytic in Δ with derivative f(z)

$\therefore f(z)dz$ is a exact differential eqn

$\therefore \int_\gamma f(z)dz = 0$ for any closed curve γ in Δ.

Lemma:

If the piecewise differential closed curve γ does not pass through the point a then the value of the integral $\int_\gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$

proof:-

$$\text{Now, } \int_\gamma \frac{dz}{z-a} = \int_\gamma d(\log(z-a))$$

$$= \int_\gamma d\log|z-a| + \int_\gamma i\, d(\arg|z-a|)$$

when z describes the closed curve $\log|z-a|$ return to its initial value and arg $(z-a)$ increases (or) decreases by a multiple of 2π.

consider the equations of γ is $z = z(t)$, $\alpha \le t \le \beta$,

Let us consider the function

$$h(t) = \int_\alpha^t \frac{z(t)}{z(t)-a}\, dt \rightarrow ①$$

Equation ① is defined and continuous on the $[\alpha, \beta]$.

Diff ① w.r. to "t"

$h'(t) = \frac{z'(t)}{z(t)-a}$, whenever $z'(t)$ is continuous

Now, the derivative of $e^{-h(t)}(z(t)-a)$ is

$$= e^{-h(t)}(-h'(t))(z(t)-a) + e^{-h(t)} z'(t)$$

$$= -e^{-h(t)} \frac{z'(t)}{z(t)-a}(z(t)-a) + e^{-h(t)} z'(t)$$

$$= -e^{-h(t)} z'(t) + e^{-h(t)} z'(t) = 0$$

$$\therefore d(e^{-h(t)}(z(t)-a)) = 0$$

Integrate,

$$e^{-h(t)}(z(t)-a) = (z(\alpha)-a) \text{ [constant]}$$

$$\therefore \frac{z(t)-a}{z(\alpha)-a} = e^{h(t)}$$

$$\therefore e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}$$

Since $z(\alpha) = z(\beta)$

$$e^{h(\beta)} = \frac{z(\beta)-a}{z(\beta)-a} = 1$$

$\therefore e^{h(\beta)} = 1 \Rightarrow h(\beta)$ must be multiple of $2\pi i$

Thus, $\int_\gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$

Hence proved.

Defn: Index point

The index of the point with respect to the curve γ is defined by the equation,

$$\eta(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$ the index is also called the winding number of γ with respect to a.

Result :-

If γ lies inside of a circle then $\eta(\gamma, a) = 0$ $\forall$ points a outside of the same circle.

Thm:

The index $\eta(\gamma, a)$ is constant in each of the regions determine by γ and 0 in the unbounded region.

proof:- Let a and b be any two points in same region determine by γ can be join by polygon which does not meet γ.

To prove : $\eta(\gamma, a) = \eta(\gamma, b)$

outside of the segment the function $\frac{(z-a)}{(z-b)}$ is never real and ≤ 0.

The principal branch of $\log\left(\frac{z-a}{z-b}\right)$ is analytic in the complement of the segment

Its derivative is equal to $\frac{1}{z-a} - \frac{1}{z-b}$

If γ does not meet the segment we have,

$$\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz = 0$$

$$\Rightarrow \int_\gamma \frac{1}{z-a} dz - \int_\gamma \frac{1}{z-b} dz = 0$$

$$\Rightarrow \int_\gamma \frac{dz}{z-a} - \int_\gamma \frac{dz}{z-b} = 0$$

$$\int_\gamma \frac{dz}{z-a} = \int \frac{dz}{z-b}$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-b}.$$

$$\Rightarrow \eta(\gamma, a) = \eta(\gamma, b)$$

If $|a|$ is sufficiently large, γ is contained in a disk $|z| < \rho < |a|$

By the result, $\eta(\gamma, a) = 0$

$\therefore \eta(\gamma, a) = 0$ is unbounded region.

Cauchy's integral formula (or) cauchy's representation

Formula:-

Suppose that $f(z)$ is analytic in an open disc Δ and let γ be a closed curve in Δ for any point a not on γ.

$$\eta(\gamma, a) f(a) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}dz$$ where

$\eta(\gamma, a)$ is the index of 'a' with respect to γ.

proof:- Let $f(z)$ be analytic in an open disk Δ.

consider, the closed curve γ in Δ and let a point 'a' which does not lie on γ

Suppose a is not in Δ

then the result is obvious.

Let $a \in \Delta$

Consider the function $F(z) = \dfrac{f(z)-f(a)}{z-a}$

This function is analytic for $z \neq a$ and not analytic for $z=a$.

$F(z)$ satisfies the condition

$$\lim_{z \to a} F(z)(z-a) = \lim_{z \to a} \frac{f(z)-f(a)}{z-a}(z-a)$$

$$= \lim_{z \to a} f(z) - f(a)$$

$$= f(a) - f(a) = 0$$

By thm,

$$\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0$$

$$= \int_\gamma \left(\frac{f(z)}{z-a} - \frac{f(a)}{z-a}\right) dz = 0$$

$$= \int_\gamma \frac{f(z)}{z-a}\,dz = \int_\gamma \frac{f(a)}{z-a}\,dz$$

$$\Rightarrow \int_\gamma \frac{f(z)}{z-a}\,dz = f(a) \int_\gamma \frac{dz}{z-a} \rightarrow (1)$$

We know that,

$$\eta(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

$$\Rightarrow 2\pi i\, \eta(\gamma, a) = \int_\gamma \frac{dz}{z-a} \rightarrow (2)$$

Sub (2) in (1)

$$\int_\gamma \frac{f(z)}{z-a}\,dz = f(a)\, 2\pi i\, \eta(\gamma, a)$$

$$\eta(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$$

If $\eta(\gamma, a) = 1$

Then, $f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$

This is known as "cauchy integral" formu pblm: 1 compute $\int_\gamma \frac{e^z}{z} dz$ where γ lies on $|z|=1$

soln:-

We know that

Cauchy's integral formula

$$\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i f(a) \rightarrow (1)$$

Given $f(z)=e^z$ and $a=0$

The point 0 lies inside $|z|=1$

$$\therefore \int_{|z|=1} \frac{e^z}{z} dz = 2\pi i f(0) \quad \left[\because 2\pi i f(a) \quad 2\pi i e^a \quad 2\pi i e^0\right]$$

$$= 2\pi i (e^0)$$

$$\int_{|z|=1} \frac{e^z}{z} dz = 2\pi i$$

pblm: 2 compute $\int_{|z|=1} e^z z^{-n} dz$

soln:-

Given $\int_{|z|=1} \frac{e^z}{z^n} dz$ Here $f(z)=e^z$, $a=0$

The point $a=0$ lies inside $|z|=1$

We know that, $\int_\gamma \frac{f(z)dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$

$$\int_{|z|=1} \frac{e^z}{z^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) \longrightarrow (1)$$

Here $f(z)=e^z$, $f'(z)=e^z \ldots f^{n-1}(z)=e^z$, $f^n(z)=e^z$

$$f^{n-1}(z)=e^z \Rightarrow f^{n-1}(0)=e^0=1$$

(1) becomes

$$\int_{|z|=1} \frac{e^z}{z^n} dz = \frac{2\pi i}{(n-1)!} \times 1 = \frac{2\pi i}{(n-1)!}$$

Pblm:
7)

$$\int_{|z|=2} \frac{z\,dz}{z^2-1}$$

Soln:-

Given $f(z) = z$

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$$

Using partial fraction

$$\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$

$$1 = A(z-1) + B(z+1)$$

$$z=1 \Rightarrow 1 = 2B, \quad B = 1/2$$

$$z=-1 \Rightarrow 1 = -2A, \quad A = -1/2$$

$$\frac{1}{z^2-1} = \frac{1}{2(z-1)} = \frac{1}{2(z+1)}$$

The point $z_0 = 1, -1$ lies inside the curve

By cauchy integral formula,

$$\int_C \frac{z}{z^2-1}\,dz = \frac{1}{2}\int\left(\frac{z}{z-1} - \frac{z}{z+1}\right)dz$$

$$= \frac{1}{2}\left[2\pi i f(1) - 2\pi i f(-1)\right]$$

$$= \frac{1}{2}(4\pi i) = 2\pi i.$$

2))

$$\int_{|z|=1} \frac{z^3}{(2z+i)^3}\,dz$$

Soln:- Given $f(z) = z^3$, $f'(z) = 3z^2$, $f''(z) = 6z$

By cauchy integral test,

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\int_{|z|=1} \frac{z^3}{2^3(z+i/2)^3}\,dz = \frac{1}{2^3}\,\frac{2\pi i}{n!}\,f^n(z_0)$$

$$= \frac{1}{2^3}\,\frac{2\pi i}{2!}\,f^n(z_0)$$

$$= \frac{\pi i}{8} \times f^{2}(-i/2)$$

$$= \frac{\pi i}{8} \times 6(-i/2) = \frac{3\pi}{8}$$

3) $\displaystyle\int_{|z|=2} \frac{e^{z}}{(z+1)^4}\, dz$

soln:- let $f(z) = e^{2z}$, $f'(z) = 2e^{2z}$, $f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f'''(-1) = 8e^{-2}$

-1 lies inside the circle.

By cauchy integral

$$f^{n}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{n}(z_0)$$

$$\int_{|z|=2} \frac{e^{2z}}{(z+1)^4}\, dz = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{2\pi i}{3!} \times 8e^{-2}$$

$$= \frac{8}{3} \times \frac{\pi i}{e^2}$$

(1) compute $\displaystyle\int_{|z|=p} \frac{|dz|}{|z-a|^2}$ under the condition $|a| \neq p$ [make use the eqn $z\bar{z} = p^3$ and $|dz| = -ip\frac{dz}{z}$]

soln:- $|z| = p$. $z = pe^{it}$; $0 \leq t \leq p$

$dz = pie^{it}\, dt$

$|dz| = |pie^{it}\, dt| \Rightarrow p|ie^{it}|dt \Rightarrow p(1)dt$

$|dz| = p\,dt \rightarrow ①$

$$dz = \rho i e^{it} dt \Rightarrow \frac{dz}{i\rho e^{it}} = dt \rightarrow (2)$$

sub (2) in (1)

$$\Rightarrow |dz| = \rho\left(-i\frac{dz}{z}\right) \Rightarrow -i\rho\frac{dz}{z}$$

$$\Rightarrow \frac{1}{|z-a|^2} = \frac{1}{(z-a)\overline{(z-a)}}$$

$$= \frac{1}{(z-a)(\bar{z}-\bar{a})}$$

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_{|z|=\rho} \frac{-i\rho dz}{z(z-a)(\bar{z}-\bar{a})}$$

$$= i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\bar{z}-\bar{a})}$$

$[\because |z|^2 = \rho^2]$

$$= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(|z|^2 - z\bar{a})}$$

$$= -i\rho \int_{|z|=\rho} \frac{\left(\frac{1}{\rho^2 - z\bar{a}}\right) dz}{z-a}$$

Here $f(z) \Rightarrow \frac{1}{\rho^2 - z\bar{a}}$

By using cauchy integral formula,

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho \frac{2\pi i f(a)}{1}$$

$$= -i\rho\, 2\pi i \frac{1}{\rho^2 - |a|^2}$$

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{2\pi\rho}{\rho^2 - |a|^2}$$

Morera's thm:

If $f(z)$ is define and continuous in a region Ω and if $\int_\gamma f(z)dz = 0\ \forall$ closed curve γ in Ω then $f(z)$ is analytic in Ω.

proof:- Given $f(z)$ is continuous in Ω and $\int_\gamma f(z)dz = 0$ for all closed curve γ in Ω

we know that

"The integral $\int_\gamma f(z)dz$ continuous depends only on the end points of $\gamma \Leftrightarrow f$ is the derivative of an analytic function in Ω" $\longrightarrow$ (1)

we know that

" The line integral depends only on the end points iff the integral is exact differential" $\longrightarrow$ (2)

$$\text{Given} \quad \int_\gamma f(z)dz = 0$$

ie) The given integral is exact

By (2) the integral depends on the end points

By (1) $f(z)$ is derivatives of an analytic function $F(z)$

$$\text{ie) } F'(z) = f(z)$$

we know that

"The derivative of an analytic function is again analytic

$\therefore f(z)$ is analytic in Ω

Hence proved.

~~Liouville's thm:-~~

2)) compute $\int_{\gamma} \frac{dz}{z^2+1}$ where γ lies on $|z|=2$

soln:-

$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$1 = A(z-i) + B(z+i)$$

put $z=i$, $1 = 2Bi \implies B = 1/2i$

$z=-i$, $1 = -2iA \implies A = -1/2i$

$$\frac{1}{z^2+1} = \frac{-1}{2i(z+i)} + \frac{1}{2i(z-i)}$$

$$= -\frac{1}{2i}\left[\frac{1}{z+i} - \frac{1}{z-i}\right]$$

$$\int_{|z|=2} \frac{dz}{z^2+1} = -\frac{1}{2i}\int_{|z|=2}\left[\frac{1}{z+i} - \frac{1}{z-i}\right]dz$$

$$= -\frac{1}{2i}\left[\int_{|z|=2}\frac{1}{z+i}dz - \int_{|z|=2}\frac{1}{z-i}dz\right]$$

The point i and $-i$ are lies inside the circle $|z|=2$

$$\int_{|z|=2}\frac{dz}{z^2+1} = -\frac{1}{2i}\left[2\pi i f(-i) - 2\pi i f(i)\right]$$

$$= -\frac{1}{2i}\left[2\pi i(1) - 2\pi i(1)\right]$$

$$= -\frac{1}{2i}(0)$$ [$\because$ By cauchy's integral formula]

$$= 0.$$

37) Evaluate $\int_\gamma \frac{e^z}{z^2+4}\,dz$ where γ is positively oriental circle $|z-i|=2$.

Soln:-

$$\frac{1}{z^2+4} = \frac{A}{z-2i} + \frac{B}{z+2i}$$

$$1 = A(z+2i) + B(z-2i)$$

put $z=-2i \Rightarrow B(-2i-2i)=1$

$$B = \frac{1}{-4i} \Rightarrow B = 4i$$

put $z=2i \Rightarrow 1=4iA \Rightarrow A = \frac{1}{4i}$

$$\frac{1}{z^2+4} = \frac{1}{4i(z-2i)} - \frac{1}{4i(z+2i)}$$

$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right)$$

$$\int_{|z-i|=2} \frac{e^z}{z^2+4}\,dz = \frac{1}{4i}\left[\int_{|z-i|=2} \frac{1}{|z-2i|}\,dz - \int_{|z-i|=2} \frac{1}{|z+2i|}\,dz\right]$$

The points $2i$ and $-2i$ are lies inside and outside of the circle $|z-i|=2$

$$\int_{|z-i|=2} \frac{e^z}{z^2+4}\,dz = \frac{1}{4i}\left[2\pi i f(2i) - 0\right]$$

$$= \frac{1}{4i}\left(2\pi i\, e^{2i}\right)$$

$$= \frac{\pi}{2} e^{2i}$$

Cauchy's higher derivative formula

$$f_n(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

pblm:1 Evaluate $\int_\gamma \frac{e^{2z}}{(z+1)^3}\,dz$ where γ is $|z|=2$.

soln:-

Given $f(z)=e^{2z}$

$a=-1$ and -1 lies inside $|z|=2$

W.K.T $\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$.

$$\int_\gamma \frac{e^{2z}}{(z+1)^3}\,dz = \frac{2\pi i}{2!} f''(-1) \quad \text{--- (1)}$$

Here $f(z) = e^{2z}$

$f'(z) = e^{2z}\cdot 2 = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f''(-1) = 4e^{-2}$

(1) becomes, $\int_\gamma \frac{e^{2z}}{(z+1)^3}\,dz = \frac{2\pi i}{2} \times 4e^{-2}$

$= 4\pi i e^{-2}$

$$\int_\gamma \frac{e^{2z}}{(z+1)^3}\,dz = \frac{4\pi i}{e^2}$$

Liovilles thm

repeated

A function which is analytic and bounded in the whole plane must reduced to a constant.

Proof:-

Let $f(z)$ is bounded and analytic in the whole plane.

Since $f(z)$ is bounded.

There exists a real number M such that $|f(z)| \le M \;\forall\; z$.

Let z_0 be any complex number and

$r>0$ be any real number.

By cauchy's inequality, we have $\frac{M\,n!}{r^n}$ $n=1$

$$|f'(z_0)| \leq \frac{M}{r}$$

Taking the limit as $r \to \infty$

$$|f'(z_0)| = 0$$

$$f'(z_0) = 0$$

Since z_0 is arbitrary

$$f'(z) = 0 \quad \forall z$$

$$\Rightarrow f(z) = \text{constant}$$

$f(z)$ is a constant function.

Fundamental theorem of Algebra

Every polynomial of degree > 0 has atleast one zero (root)

proof:- Let $p(z)$ be the polynomial of degree >0

suppose $p(z)$ has no zero.

Then $p(z) \neq 0 \quad \forall z$.

$p(z)$ is the entire function in the whole plane.

$p(z)$ is analytic in the whole plane.

$1/p(z)$ is analytic in the whole plane.

As $z \to \infty$, $p(z) \to \infty$

$$\frac{1}{p(z)} \to 0 \quad \text{as } z \to \infty$$

$\frac{1}{p(z)}$ is the bounded function.

$\frac{1}{p(z)}$ is analytic and bounded

[by Liouville's thm]

$\Rightarrow \frac{1}{p(z)}$ is contant

$p(z)$ is constant function.

$p(z)$ is a polynomial of degree zero which is a contradiction to $p(z)$ is a polynomial of deg >0.

$\therefore p(z)$ has atleast one zero

Hence proved.

Cauchy's inequality (or) Estimate

Let $f(z)$ be analytic inside on the circle with centre z_0 and radius r. Let M denote the maximum of $|f(z)|$ on c then $|f^n(z_0)| \leq \frac{n!\,m}{r^n}$.

proof:-

W.K.T, the higher derivatives of cauchy integral formula is

$$f^n(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

$$|f^n(z_0)| = \left|\frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

$$\leq \frac{n!}{2\pi i}\int_c \frac{|f(z)|}{|z-z_0|^{n+1}}\,|dz|$$

Given $|f(z)| \leq M$ and $|z-z_0| \leq r$

$$|f^n(z_0)| \leq \frac{n!}{2\pi}\int_c \frac{M}{r^{n+1}}\,|dz|$$

$$= \frac{n!}{2\pi}\left(\frac{M}{r^{n+1}}\right)\int_c |dz|$$

$$= \frac{n!}{2\pi}\,\frac{M}{r^{n+1}}\,2\pi r.$$

$$|f^n(z_0)| \leq \frac{n!\,M}{r^n}$$

Hence proved

Thm suppose that $\phi(\zeta)$ is continuous on the

10m (3) arc γ the function $F_n(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n} d\zeta$

is analytic in each of the regions determine by γ and its derivative is $F_n'(z) = n F_{n+1}(z)$.

proof:- First we have to prove

$F_1(z)$ is continuous

Let z_0 be a point not in γ.

Let us choose a neighbourhood $|\zeta - z| < \delta$,

$|z - z_0| < \delta$.

Now, restrict the point z to smaller that

$|z - z_0| < \delta/2$ for all $\zeta \in \gamma$, $|\zeta - z| > \delta/2$

Now, $F_1(z) - F_1(z_0) = \int_\gamma \frac{\phi(\zeta)}{\zeta - z} d\zeta - \int_\gamma \frac{\phi(\zeta)}{\zeta - z_0} d\zeta$

$$= \int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) \phi(\zeta) d\zeta$$

$$= \int_\gamma \frac{\zeta - z_0 - \zeta + z}{(\zeta - z)(\zeta - z_0)} \phi(\zeta) d\zeta$$

$$= \int_\gamma \frac{z - z_0}{(\zeta - z)(\zeta - z_0)} \phi(\zeta) d\zeta$$

$$F_1(z) - F_1(z_0) = (z - z_0) \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)} d\zeta \rightarrow (1)$$

$$|F_1(z) - F_1(z_0)| \leq |z - z_0| \int_\gamma \frac{|\phi(\zeta)| \, |d\zeta|}{|\zeta - z| \, |\zeta - z_0|}$$

$$< |z - z_0| \cdot \frac{2}{\delta} \cdot \frac{1}{\delta} \int_\gamma |\phi(\zeta)| |d\zeta|$$

If $|\phi(\zeta)| \leq M$ and $\int_\gamma |d\zeta| = l$ (length of the arc)

$$\therefore |F_1(z)-F_1(z_0)| \le \frac{2Ml}{\delta^2}|z-z_0|$$

$$\lim_{z\to z_0}|F_1(z)-F_1(z_0)| \le \frac{2Ml}{\delta^2}\lim_{z\to z_0}|z-z_0|$$

$$\lim_{z\to z_0}|F_1(z)-F_1(z_0)|=0$$

$$\lim_{z\to z_0}F_1(z)=F_1(z_0)$$

$\therefore$ $F_1(z)$ is continuous at z_0

$\Rightarrow$ $F_1(z)$ is analytic at z_0.

From ① $$\lim_{z\to z_0}\frac{F_1(z)-F_1(z_0)}{z-z_0} = \lim_{z\to z_0}\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)(\zeta-z_0)}$$

$$F_1'(z_0)=\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)(\zeta-z_0)}$$

$$=\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^2}$$

$F_1'(z_0)-F_2(z_0)$ ie) $F_1'(z)=F_2(z)$

Hence the theorem is true for $n=1$.

Let us assume that the theorem is true for $n-1$.

ie) $F_{n-1}(z)$ is analytic and $F_{n-1}(z)=(n-1)F_n(z)$

We shall prove that the theorem is true for n.

$F_n(z)$ is analytic and $F_n'(z)=nF_{n+1}(z)$

Now,

$$\frac{1}{\zeta-z}-\frac{1}{\zeta-z_0}=\frac{\zeta-z_0-\zeta+z}{(\zeta-z)(\zeta-z_0)}=\frac{z-z_0}{(\zeta-z)(\zeta-z_0)} \rightarrow ②$$

Let us consider

$$\left[\frac{1}{(\zeta-z)^{n-1}}+\frac{1}{(\zeta-z_0)^{n-1}}\right]\left[\frac{1}{\zeta-z}-\frac{1}{\zeta-z_0}\right]=\left[\frac{1}{(\zeta-z)^{n-1}}+\frac{1}{(\zeta-z_0)^{n-1}}\right]\left[\frac{z-z_0}{(\zeta-z)(\zeta-z_0)}\right]$$

$$\Rightarrow \frac{1}{(\zeta-z)^n} - \frac{1}{(\zeta-z_0)(\zeta-z)^{n-1}} + \frac{1}{(\zeta-z)(\zeta-z_0)^{n-1}} - \frac{1}{(\zeta-z_0)^n}$$

$$= (z-z_0)\left[\frac{1}{(\zeta-z_0)(\zeta-z)^n} + \frac{1}{(\zeta-z)(\zeta-z_0)^n}\right]$$

$$\Rightarrow \frac{1}{(\zeta-z)^n} - \frac{1}{(\zeta-z_0)^n} = \frac{1}{(\zeta-z_0)(\zeta-z)^{n-1}} - \frac{1}{(\zeta-z)(\zeta-z_0)^{n-1}}$$

$$+(z-z_0)\left[\frac{1}{(\zeta-z_0)(\zeta-z_0)^n} + \frac{1}{(\zeta-z)(\zeta-z_0)^n}\right]$$

$$= \frac{1}{(\zeta-z_0)(\zeta-z)^{n-1}} + \frac{z-z_0}{(\zeta-z)^n(\zeta-z_0)} + \left[\frac{z-z_0}{(\zeta-z)(\zeta-z_0)^n} - \frac{1}{(\zeta-z_0)^{n-1}(\zeta-z)}\right]$$

$$= \frac{1}{(\zeta-z)^{n-1}(\zeta-z_0)} + \frac{z-z_0}{(\zeta-z)^n(\zeta-z_0)} + \left[\frac{z-z_0-\zeta+z_0}{(\zeta-z_0)^n(\zeta-z)}\right]$$

$$= \frac{1}{(\zeta-z)^{n-1}(\zeta-z_0)} + \frac{z-z_0}{(\zeta-z)^n(\zeta-z_0)} + \frac{z-\zeta}{(\zeta-z_0)^n(\zeta-z)}$$

$$= \frac{1}{(\zeta-z)^{n-1}(\zeta-z_0)} + \frac{z-z_0}{(\zeta-z)^n(\zeta-z_0)} - \frac{(\zeta-z)}{(\zeta-z_0)^n(\zeta-z)}$$

$$= \frac{1}{(\zeta-z)^{n-1}(\zeta-z_0)} + \frac{z-z_0}{(\zeta-z)^n(\zeta-z_0)} - \frac{1}{(\zeta-z_0)^n}$$

Multiplying by $\phi(\zeta)$ and integrate with respect to ζ on γ.

$$\int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^n}\,d\zeta - \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^n} = \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^{n-1}(\zeta-z_0)}$$

$$+(z-z_0)\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^n(\zeta-z_0)} - \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^n}$$

$$\Rightarrow \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^n} - \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^n} = \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^{n-1}(\zeta-z_0)}$$

$$- \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^n} + (z-z_0)\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^n(\zeta-z_0)} \longrightarrow (3)$$

$$\lim_{z\to z_0} F_n(z)-F_n(z_0) = \lim_{z\to z_0} \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^n} - \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^n}$$

$$+ \lim_{z\to z_0} \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}$$

$$\lim_{z\to z_0} F_n(z)-F_n(z_0) = 0$$

$$\Rightarrow \lim_{z\to z_0} F_n(z) = F_n(z_0)$$

$F_n(z)$ is continuous.

$\therefore F_n(z)$ is analytic

claim: $F_n'(z) = nF_{n+1}(z)$

Let us define,

$$g_{n-1}(z) = \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^{n-1}(\zeta-z_0)}$$

Now, $$\lim_{z\to z_0} \frac{g_{n-1}(z)-g_{n-1}(z_0)}{z-z_0} = g_{n-1}'(z_0)$$

$$= (n-1)g_n(z_0)$$

[Since, $g_n(z_0) = \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^n(\zeta-z_0)}$]

$$= (n-1)\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}$$

$$\lim_{z\to z_0} \frac{g_{n-1}(z)-g_{n-1}(z_0)}{z-z_0} = (n-1)F_{n+1}(z_0) \longrightarrow (4)$$

From (3),

$$F_n(z)-F_n(z_0) = g_{n-1}(z) - g_{n-1}(z_0) + (z-z_0)\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^n(\zeta-z_0)}$$

$$\lim_{z \to z_0} \frac{F_n(z) - F_n(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{g_{n-1}(z) - g_{n-1}(z_0)}{z - z_0}$$

$$+ \lim_{z \to z_0} \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi - z)^n (\xi - z_0)}$$

$$F_n'(z_0) = (n-1) F_{n+1}(z_0) + F_{n+1}(z_0)$$

By using eqn (4)

$$\Rightarrow F_n'(z_0) = n F_{n+1}(z_0) - F_{n+1}(z_0) + F_{n+1}(z_0)$$

$$= n F_{n+1}(z_0)$$

$$\therefore F_n'(z_0) = n F_{n+1}(z_0)$$

Hence proved.

Unit - III

* Local properties of analytic functions

Unit-III

Local properties of analytic function

Singular point:

A point 'a' is said to be a singular point or singularity of a function $f(z)$ if $f(z)$ is not analytic at 'a' and $f(z)$ is analytic at some point of every disk $|z-a| < r$.

Eg:

consider the function $f(z) = \frac{1}{z(z-i)}$, 0 and i are singular points for $f(z)$.

Isolated singularity:

A point 'a' is called an isolated singularity for $f(z)$ if, $f(z)$ is not analytic at $z=a$ and there exist $r>0$ such that $f(z)$ is analytic in $0 < |z-a| < r$.

ie) The neighbourhood $|z-a| < r$ contains no singularity of $f(z)$ except "a".

Eg:
$f(z) = \dfrac{z+1}{z^2(z^2+1)}$ has three isolated singularities

$z = 0, i, -i$

Isolated singularity

Removable singularity | pole | Essential singularity.

Thm:

Suppose that $f(z)$ is analytic in the region Ω' obtain by omitting a point 'a' from a region Ω. A necessary and sufficient condition that there exist an analytic function in Ω which coincides with $f(z)$ in Ω' is that $\lim_{z \to a} (z-a) f(z) = 0$ the extended function is uniquely determined.

proof:-

The necessity and uniqueness are trivial. Since the extended function must be continuous at a.

To prove:

The sufficiency.

We draw a circle 'c' about 'a' so that c an inside and contained in Ω. By cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)}$$ for all $z \neq a$

inside of c.

But the integral in the right hand numbers represents an analytic function of z through out of inside of c

consequently,

the function which is equal to $f(z)$ for $z \neq a$,

and which has the value.

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - a)} \quad \text{—} ①$$

where we denote the extended function by $f(z)$ and the value ① by $f(a)$

We apply this result to the function,

$$f(z) = \frac{f(z) - f(a)}{z - a} \quad \text{—} ②$$

Equation ② is not define for $z = a$.

But it satisfies the condition

$\lim_{z \to a} (z-a) f(z) = 0$.

The limit of $f(z)$ as $z \to a$ is $f'(a)$

Hence there exists an analytic function which is equal to $F(z)$ for $z \neq a$ and equal to $f'(z)$ for $z = a$.

10m Taylor's thm

(*) If $f(z)$ is analytic in a region Ω containing in 'a' it is possible to wide

$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + \frac{f_n(a)}{n!}(z-a)^n$ where $f_n(a)$ is analytic in Ω.

proof:- consider the function $F(z) = \frac{f(z) - f(a)}{z-a}$

It is analytic at $z \neq a$ and for $z = a$

It is not analytic but it satisfies

$\lim_{z \to a} (z-a) f(z) = 0$

F can be re-define in such a way that it becomes analytic at $z = a$.

$$\lim_{z \to a} F(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z-a} = f'(a)$$

$$f_1(z) = \begin{cases} \frac{f(z) - f(a)}{z-a} & \forall \ z \neq a \\ f'(a) & \text{at } z = a \end{cases} \quad —①$$

$f_1(z)$ is analytic in Ω

$$f_2(z) = \begin{cases} \frac{f_1(z) - f_1(a)}{z-a} & \forall \ z \neq a \\ f_1'(a) & \text{at } z = a \end{cases} \quad —②$$

$f_2(z)$ is analytic in Ω

proceeding like this we get,

$$f_n(z) = \begin{cases} \frac{f_{n-1}(z) - f_{n-1}(a)}{z-a} & \forall \ z \neq a \\ f_{n-1}'(a) & \text{at } z = a \end{cases}$$

which is analytic in Ω

For $z \neq a$

From the defn of function.

$$\begin{aligned} f(z) &= f(a) + (z-a) f_1(z) \quad [\text{by } ①] \\ &= f(a) + (z-a)[f_1(a) + (z-a) f_2(z)] [\text{by} ②] \\ &= f(a) + (z-a) f_1(a) + (z-a)^2 [f_2(a) + (z-a) f_3(z)] \\ &= f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + (z-a)^3 f_3(z) \end{aligned}$$

preceeding like this we get

$$f(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + \dots\dots + (z-a)^{n-1} f_{n-1}(a) + (z-a)^n f_n(z) \quad \text{--- (3)}$$

Diff w.r.to z, we get

$$f'(z) = f_1(a) + 2(z-a) f_2(a) + 3(z-a)^2 f_3(a) + \dots$$

$$f'(a) = f_1(a) + 0 + 0$$

$$f'(a) = f_1(a)$$

lll^{ly}, $f''(z) = 2 f_2(a) + 6(z-a) f_3(a) + \dots$

$$f''(a) = 2 f_2(a)$$

$$f''(a) = 2! f_2(a)$$

lll^{ly}, $f'''(z) = 6 f_3(a)$

$$= 3! f_3(a)$$

In general $f^n(a) = n! f_n(a)$

$$\Rightarrow f_n(a) = \frac{f^n(a)}{n!} \quad \forall n$$

sub these values in (3),

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2 \frac{f''(a)}{2!} + \dots + (z-a)^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!} + (z-a)^n \frac{f_n(z)}{n!}$$

Hence proved.

Meromorphic function

Prove that the poles of meromorphic function are isolated.

A function $f(z)$ which is analytic in a region Ω, except for poles is said to be meromorphic in Ω.

More precisely, to every $a \in \Omega$, there shall exist a neighbourhood, $|z-a|<\delta$ contained in Ω such that either $f(z)$ is analytic in the whole neighbourhood or else $f(z)$ is analytic for $0<|z-a|<\delta$ ~~also~~ and the isolated singularity is a pole

By the defn the poles of meromorphic function are isolated.

Ex:

$$f(z) = \frac{1}{z(z-1)^2}$$

Here 0 and 1 are poles of order 1 and 2 respectively.

The function $f(z)$ is analytic except the poles 0 and 1.

$\therefore$ $f(z)$ is meromorphic function.

Thm: Weirstrass thm

An analytic function comes arbitrary closed to any complex value in every neighbourhood of an essential singularity.

proof:-

Let f be analytic except at a where a is an essential singularity for f.

Suppose the theorem is not true.

ie) there exists a complex number A and δ such that $f(z)$ is not closed to A in a neighbourhood of a.

ie) $|f(z)-A| > \delta$ for any $\alpha < 0$

We have, $\lim_{z \to a} |z-a|^{\alpha} |f(z)-A| = \infty$

Here a is not an essential singularity of $f(z) \to A$.

choose $\beta > 0$

$$\lim_{z \to a} |z-a|^{\beta} |f(z)-A| = 0$$

since $$\lim_{z \to a} |z-a|^{\beta} |f(z)| = 0$$

$$\lim_{z \to a} |z-a|^{\beta} |A| = 0$$

$\therefore$ a is not an essential singularity of $f(z)$ which is contradiction.

$\therefore$ f is arbitrary close to any complex number in every neighbourhood of A.

(24) problem

Show that the function e^z, $\sin z$, $\cos z$ have essential singularity at ∞.

soln:- Let $f(z) = e^z$, $f(1/z) = e^{1/z}$ which is analytic except at $z=0$.

$$f(1/z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \cdots$$

$$= 1 + \frac{1}{z\,1!} + \frac{1}{z^2\,2!} + \cdots$$

where the principal part of contains infinitely many terms.

$z=0$ is an essential singularity for $f(1/z)$

$z=\infty$ is an essential singularity for $f(x) = e^x$.

Local mapping theorem

Let z_j be the zeros of the function $f(z)$ which is analytic in a disk Δ and does not vanish identically, each zero being counted as many times as its order indicates, for every closed curve γ in Δ which does not pass through a zero $\sum_{j}^{n} \eta(\gamma, z_j) = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz$ where the sum has only a finite number of terms not equal to zero.

proof:

case(i)

Let $z_1, z_2, \ldots, z_n$ be the finite numbers of zeroes in Δ.

Then $f(z) = [(z-z_1)(z-z_2)\ldots(z-z_n)\phi(z)]$ where $\phi(z)$ is analytic and not equal to zero in Δ.

$$\log f(z) = \log[(z-z_1)(z-z_2)\ldots(z-z_n)\phi(z)]$$
$$= \log(z-z_1) + \log(z-z_2) + \ldots + \log(z-z_n) + \log\phi(z)$$
$$\log f(z) = \sum_{j=1}^{n} \log(z-z_j) + \log\phi(z)$$

Diff w.r. to z,

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z-z_j} + \frac{\phi'(z)}{\phi(z)}$$

multiply both sides by $1/2\pi i$ and integrating over any closed curve γ in Δ that does not pass through any of z_j's, we get,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_j} + \frac{1}{2\pi i}\int_\gamma \frac{\phi'(z)}{\phi(z)}\,dz$$

$$= \sum_{j=1}^{n} \eta(\gamma, z_j) + \frac{1}{2\pi i}\int_\gamma \frac{\phi'(z)}{\phi(z)}\,dz \quad \text{--- (1)}$$

$\therefore$ $\phi(z)$ is analytic in Δ.

$\phi'(z)$ is also analytic in Δ.

$\therefore$ $\dfrac{\phi'(z)}{\phi(z)}$ is analytic in Δ

By cauchy's theorem is an open disc

$$\int_\gamma \frac{\phi'(z)}{\phi(z)}\,dz = 0$$

The above eqn becomes

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \eta(\gamma, z_j) \quad //$$

case(ii)

Suppose that f has infinitely many zeroes in Δ, choose a concentric disk Δ'.

So that,

Δ' contains the closed curve γ.

If Δ' encloses infinitely many zero's of f.

Then the infinite set of zero's have a limit point in Δ'.

which is impossible.

Δ' have only finite number of zero of f.

Let α_j' denote the zero's of f in Δ' and let α_j denote the zero's outside.

Since γ lies inside Δ' and α_j's outside Δ'

$$\eta(\gamma, \gamma_j) = 0 \quad \forall j$$

Since Δ' encloses only a finite number as zero's of f.

By case (i)

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j \eta(\gamma, \gamma_j')$$

$$= \sum_j \eta(\gamma, \gamma_j') + \sum_j \eta(\gamma, \gamma_j)$$

$$= \sum_j \eta(\gamma, z_j)$$ which has only finite no of non-zero terms

$$\sum_j \eta(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

(x) ~~Local mapping theorem or~~ local correspond thm

Suppose that $f(z)$ is analytic at a point z_0 and $f(z_0) = w_0$ and that $f(z) - w_0$ has a zero of order n at z_0.

If $\varepsilon > 0$ is sufficiently small there exists a corresponding $\delta > 0$ such that for all a with $|a - w_0| < \delta$. The equation $f(z) = a$ has exact n roots in the disk $|z - z_0| < \varepsilon$.

proof:- choose $\varepsilon > 0$ sufficiently small such that f is differentiable in $|z-z_0| < \varepsilon$

And z_0 is the only zero as $f(z)-w_0$ in this neighbourhood.

Let γ denote the boundary of this neighbourhood.

ie) γ denotes $|z-z_0| = \varepsilon$.

Let Γ be the image of γ under f. Since z_0 is an interior point of γ whose image is w_0 under f.

$w_0 \in \Gamma^c$.

Since Γ is closed its complement Γ^c is open.

there exists a neighbourhood $|w-w_0| < \delta$ contained in Γ^c

ie) $|w-w_0| < \delta \subseteq \Gamma$

ie) The neighbourhood $|w-w_0| < \delta$ does not intersect Γ.

Also $f(z)$ takes the value w_0 in this neighbourhood n times

By corollary,

$f(z)$ takes every value in $|w-w_0| < \delta$, n times.

ie) The equation $f(z)=a$ has n roots in $|z-z_0| < \varepsilon$ for every a such that $|a-w_0| < \delta$

Note:

a and b are in same region determine by Γ.

W.K.T, $\eta(\Gamma, a) = \eta(\Gamma, b)$ and hence $\sum_j \eta(\gamma, z_j(a)) = \sum_j \eta(\gamma, z_j(b))$

If γ is a circle $f(z)$ takes a value a and b equally many times inside of γ.

Corollary: 1

A non-constant analytic function maps open sets onto open sets.

proof:-

Let $f(z)$ be analytic function in Δ.

To prove,

By the local correspondence thm,

For case $n=1$

There is 1-1 correspondence between disk $|w-w_0| < \delta$ and the open subset Δ of $|z-z_0| < \varepsilon$.

W.K.T, "Inverse function of $f(z)$ is constant and f is constant iff converse image of open sets is open".

The open set in the z-plane corresponds to open set in w-plane.

Hence f maps open sets onto open sets.

Maximum principle or maximum modulus theorem

i) If $f(z)$ is analytic and non-constant in a region Ω, then its absolute value $|f(z)|$ has no maximum in Ω.

ii) If $f(z)$ is define and constant on a closed and bounded set E and analytic on the interior of E, then the max of $|f(z)|$ on E is assume on the boundary of E.

proof:-

i) Let w_0 be a value taken by f in Ω.

Since Ω is open and f is a non-constant analytic function.

$\therefore f(\Omega)$ is open.

Also $w_0 \in f(\Omega)$, there exist a neighbourhood $|w-w_0|<\delta$ which is fully contained in $f(\Omega)$.

In this neighbourhood there are points w such that $|w|>|w_0|$.

$\therefore |w_0|$ is not the maximum of $|f(z)|$.

$\therefore |f(z)|$ has no maximum in Ω.

ii) Since f is analytic on the closed and bounded set E which is compact.

$|f(z)|$ has max of E.

Suppose that it is assume at z_0

If z_0 is on the boundary then there is nothing to prove.

If z_0 is interior point

Then $|f(z_0)| = \max |f(z)|$

since z_0 is an interior point of E

there exists a neighbourhood $|z-z_0|<\epsilon \subset E$

In this neighbourhood f is an analytic function and $|f(z_0)|$ is the max of $|f(z)|$.

which is contradiction to (i)

$\therefore$ maximum of $|f(z)|$ is obtained only at the boundary of E.

Schwartz lemma

10m.

If $f(z)$ is analytic for $|z|<1$ and satisfy the condition $|z|<1$, $f(0)=0$ then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. If $|f(z)| = |z|$ for some $z \ne 0$ (or) if $|f'(0)|=1$. Then $f(z)=cz$ with a constant c of absolute value 1.

proof: consider the function

$$f_1(z) = \begin{cases} \frac{f(z)}{z} & \forall\ z \ne 0 \text{ on the circle } |z| \\ f'(0) & \text{at } z=0 \end{cases}$$

Then $|f_1(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} \le \frac{1}{r}$

As $r \to 1$, $|f_1(z)| \le 1$

$$\Rightarrow \left|\frac{f(z)}{z}\right| \le 1 \Rightarrow \frac{|f(z)|}{|z|} \le 1$$

$$\Rightarrow |f(z)| \le |z|$$

$$\Rightarrow |f'(z)| \le |1| = 1$$

$\Rightarrow |f'(z)| \leq 1$

$\Rightarrow |f'(0)| \leq 1$

suppose the equality holds at a single point. Then $|f_1(z)|$ attains its maximum and hence $f_1(z)$ must reduces to a constant

$\therefore f_1(z) = c$

ie) $\frac{f(z)}{z} = c$

~~$f(z) = cz \Rightarrow c = \frac{f(z)}{z}$~~

$$|c| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|}$$

$f(z) = cz$ with constant c of abosule value 1.

Line integral problem

1). P.T $\int_C \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \neq 1 \text{ where c} \\ 2\pi i & \text{if } n = 1 \end{cases}$ ~~is the circle with~~

is the circle with centere a and radius r and $n \in z$.

soln:

The parameteric equation of the circle c is given by $z - a = re^{it}$, $0 \leq t \leq 2\pi$

$\therefore z'(t) = ire^{it}$

Now, $\int_C \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{it}}{(re^{it})^n} dt$

$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt$

$= \frac{i}{r^{n-1}} \left[\frac{e^{i(1-n)t}}{i(1-n)}\right]_0^{2\pi}$ provided $n \neq 1$

$$= \frac{1}{(1-n)r^{n-1}} \left[e^{i(1-n)2\pi} - 1 \right]$$

$$= \frac{1}{(1-n)r^{n-1}} (1-1)$$

$$= 0$$

If $n=1$, $\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt$

$$= i \int_0^{2\pi} dt$$

$$= 2\pi i.$$

Hence the result.

2) Evaluate $\int_C \bar{z}\, dz$ from $z=0$ to $z=4+2i$ along the curve c consists of the line segment from $z=0$ to $z=2i$ followed by the line segment from $z=2i$ to $z=4+2i$

Soln:- Let c_1 denote the line segment joins 0 to $2i$ and c_2 denote the line segment joins $2i$ to $4+2i$.

Then $c = c_1 + c_2$

Now, the parametric eqn of c_1 is given by $x(t)=0$ and $y(t)=t$ where $0 \le t \le 2$.

Hence $z(t) = x(t) + iy(t) = it$

So that $z'(t) = i$

Hence $\int_C \bar{z}\, dz = \int_0^2 (-it)\, i\, dt$

$$= \int_0^2 t\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^2$$

$$= 4/2$$

$$= 2.$$

Now the parametric eqn of c_2 is given by $x(t) = t$ and $y(t) = 2$ where $0 \le t \le 4$.

Hence $z(t) = t + 2i$ and

$$z'(t) = 1$$

$$\int_{c_2} \bar{z}\, dz = \int_0^4 (t - 2i)\, dt$$

$$= \left[\frac{t^2}{2} - 2it\right]_0^4$$

$$= \frac{16}{2} - 2i(4)$$

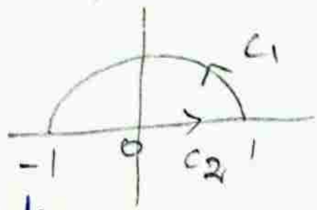
$$= 8 - 8i$$

$$\int_c \bar{z}\, dz = \int_{c_1} \bar{z}\, dz + \int_{c_2} \bar{z}\, dz$$

$$= 2 + 8 - 8i$$

$$= 10 - 8i.$$

3) Evaluate $\int_c |z|\bar{z}\, dz$ where c is the closed curve consisting of the upper semicircle $|z| = 1$ and the segment $-1 \le x \le 1$.



Soln:
Let $f(z) = |z|\bar{z}$

$$\int_c f(z)\, dz = \int_{c_1} f(z)\, dz + \int_{c_2} f(z)\, dz$$

where c_1 is the upper semicircle $|z| = 1$ and c_2 is the line segment $-1 \le x \le 1$

The parametric eqn of c_1 is given by $z = e^{it}$, $0 \le t \le \pi$

Hence $z'(t) = ie^{it}$ [$\because |z| = 1$, $\bar{z} = e^{-it}$, $dz = ie^{it} dt$]

$$\int_{C_1} f(z)\, dz = \int_0^{\pi} e^{-it} ie^{it}\, dt$$

$$= i \int_0^{\pi} dt$$

$$= \pi i$$

The parametric eqn of c_2 is grn by $y = 0$, $x = t$ where $-1 < t \le 1$.

Hence $z(t) = t$ and $z'(t) = 1$

Also $|z(t)| = \begin{cases} -t & \text{if } -1 \le t \le 0 \\ t & \text{if } 0 < t \le 1 \end{cases}$

Hence $\displaystyle\int_{C_2} |z| \bar{z}\, dz = \int_{-1}^{0} -t \cdot t\, dt + \int_0^1 t \cdot t\, dt$

$$= \left[\frac{-t^3}{3}\right]_{-1}^{0} + \left[\frac{t^3}{3}\right]_0^1$$

$$= -1/3 + 1/3$$

$$= 0$$

Hence

$$\int_C |z| \bar{z}\, dz = \int_{C_1} |z| \bar{z}\, dz + \int_{C_2} |z| \bar{z}\, dz$$

$$= \pi i$$

4) prove $\displaystyle\int_C \bar{z}^2 dz = \begin{cases} 0 & \text{if } c \text{ is the unit circle } |z| = 1 \\ 4\pi i & \text{if } c \text{ is the circle } |z-1| = 1 \end{cases}$

soln:-

Let c be the unit circle $|z| = 1$.

The parametric eqn of c is grn by $z(t) = e^{it}$ where $0 < t \le 2\pi$.

Hence $z'(t) = ie^{it}$ and

$[\bar{z}(t)]^2 = e^{-2it}$ $\quad$ $[\, z = e^{it}$

$\bar{z} = e^{-it}$

$(\bar{z})^2 = (e^{-it})^2$

$= e^{-2it} \,]$

$$\therefore \int_c \bar{z}^2 \, dz = \int_0^{2\pi} [\bar{z}(t)]^2 \, z'(t)\, dt$$

$$= i \int_0^{2\pi} e^{-it} \, dt$$

$$= -\left[e^{-it}\right]_0^{2\pi}$$

$$= 0$$

Now let c be the circle $|z-1| = 1$

The parametric eqn of c is given by

$z(t) = 1 + e^{it}$ where $0 \le t \le 2\pi$

Hence $z'(t) = ie^{it}$

$$\int_c \bar{z}^2 \, dz = \int_0^{2\pi} (1+e^{-it})^2 \, ie^{it} \, dt$$

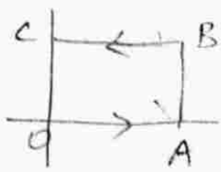
$$= i \int_0^{2\pi} (e^{it} + e^{-it} + 2) dt$$

$$= i \left[\frac{e^{it}}{i} - \frac{e^{-it}}{i} + 2t \right]_0^{2\pi}$$

$$= \left[e^{it} - e^{-it} + 2it\right]_0^{2\pi}$$

$$= 4\pi i.$$

5) S.T $\int_c |z|^2 dz = -1+i$ where c is the square with vertices $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$



soln:-

$c = c_1 + c_2 + c_3 + c_4$

where c_1, c_2, c_3 and c_4 are the line segments OA, OB, BC and CO as shown

The parametric equation of c_1 is given by $x = t$ and $y = 0$ where $0 \le t \le 1$.

Hence $z(t) = t$ and $z'(t) = 1$.

$$\therefore \int_{c_1} |z|^2 \, dz = \int_0^1 t^2 \, dt = 1/3$$

The parametric equation of c_2 is given by $y = t$ and $x = 1$ where $0 \le t \le 1$.

Hence $z(t) = 1 + it$

$z'(t) = i$

$$\int_{c_2} |z|^2 \, dz = \int_0^1 |1 + it|^2 \, dz$$

$$= i \int_0^1 (1 + t^2) \, dt$$

$$= i \left[t + \frac{t^3}{3} \right]_0^1$$

$$= \frac{4i}{3}$$

The parametric eqn of c_3 is given by $y = 1$ and $x = 1 - t$, $0 \le t \le 1$.

Hence $z(t) = (1 - t) + i$

$z'(t) = -1$

$$\therefore \int_{c_3} |z|^2 \, dz = \int_0^1 \left[(1-t)^2 + 1 \right] (-1) \, dt$$

$$= - \int_0^1 (t^2 - 2t + 2) \, dt$$

$$= - \left(t^3/3 - 2\,t^2/2 + 2t \right)_0^1$$

$$= - \left(1/3 - 2/2 + 2 \right)$$

$$= - \left(1/3 + 1 \right)$$

$$= - 4/3$$

The parametric eqn of C_4 is given by $x=0,\ y=1-t,\ 0\le t\le 1$.

Hence $z(t) = i(1-t)$ and

$$z'(t) = -i$$

$$\int_{C_4} |z|^2 dz = \int_0^1 (1-t)^2 (-i)\, dt$$

$$= -i \left[\frac{(1-t)^3}{3}\right]_0^1$$

$$= -i\left[0 - \tfrac{1}{3}\right]$$

$$= i/3$$

Hence $\int_C f(z)\,dz = 1/3 + 4i/3 - 4/3 - i/3$

$$= -1+i$$

6) Evaluate $\int_C \frac{z+2}{z}\,dz$ where c is the semi circle $z = 2e^{i\theta}$ where $0\le\theta\le\pi$.

Soln:- Here $z'(\theta) = 2ie^{i\theta}$

So that $dz = 2ie^{i\theta}\,d\theta$

$$\int_C \frac{z+2}{z}\,dz = \int_0^{\pi} \left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right)(2ie^{i\theta}\,d\theta)$$

$$= 2i\int_0^{\pi} (1+e^{i\theta})\,d\theta$$

$$= 2i\left[\theta + \frac{e^{i\theta}}{i}\right]_0^{\pi}$$

$$= 2i\left[(\pi - 1/i) - (1/i)\right]$$

$$= 2i\left[\pi - 2/i\right]$$

$$= 2i\left(\frac{\pi i - 2}{i}\right)$$

$$= -4 + 2\pi i.$$

7) Evaluate the integral $\int_c (x^2-iy^2)\,dz$ where c is the parabola $y=2x^2$ from $(1,2)$ to $(2,8)$.

Soln:-

Let $f(z)=x^2-iy^2$

The parametric eqn of c is given by $x=t$ and $y=2t^2$ where $1\le t\le 2$.

$$z(t)=x(t)+iy(t)=t+i2t^2$$

$$z'(t)=1+4it$$

$$\int_c (x^2-iy^2)\,dz=\int_1^2 (t^2-4it^4)(1+4it)\,dt$$

$$=\int_1^2 \left[(t^2+16t^5)+i(4t^3-4t^4)\right]dt$$

$$=\left[\left(\frac{t^3}{3}+\frac{16t^6}{6}\right)+i\left(t^4-\frac{4t^5}{5}\right)\right]_1^2$$

$$=\frac{8}{3}+\frac{1024}{6}+i\left(16-\frac{128}{5}\right)-\frac{1}{3}-\frac{16}{6}-i\left(1-\frac{4}{5}\right)$$

$$=\frac{7}{3}+\frac{1008}{6}+i\left(16-\frac{128}{5}-1+\frac{4}{5}\right)$$

$$=\frac{7}{3}+\frac{1008}{6}+i\left(15-\frac{124}{5}\right)$$

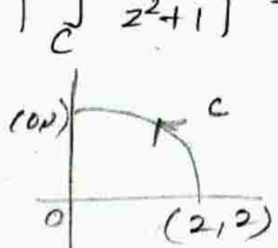
$$=\frac{7}{3}+168+i\left(\frac{75-124}{5}\right)$$

$$=\frac{7+504}{3}+i\left(\frac{-49}{5}\right)$$

$$\int_c (x^2-iy^2)\,dz=\frac{511}{3}-i\frac{49}{5}$$

8) Let c be the arc of circle $|z|=2$ from $z=2$ to $z=2i$ that lies in the first quadrant without actually evalitines the integral show that $\left|\int_C \frac{dz}{z^2+1}\right| \le \pi/3$

soln:-



Let $f(z) = \frac{1}{z^2+1}$

Since c is the circular arc of radius 2 lying in the first quadrant the length l of c is given by $l = \frac{1}{4}(2\pi \times 2) = \pi$

Also on c,

$$|z^2+1| = |z^2-(-1)|$$
$$\ge |z^2| - |-1|$$
$$= |z|^2 - 1 = 3$$

Thus $|z^2+1| \ge 3$

$$\therefore \left|\frac{1}{z^2+1}\right| \le 1/3$$

Hence by thm 6.2, ($|\int_C f(z)dz| = ml$ where $m = \max\{|f(z)| / z \in c\}$ and l is the length of c)

$$\left|\int_C \frac{dz}{z^2+1}\right| \le \pi/3.$$

9) Without evaluating the integral show that $\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$ where c is the line segment from $z=i$ to $z=1$.

soln:-
c is the line segment joins $(0,1)$ to $(1,0)$ and its length is obviously $\sqrt{2}$.

As z varies on c, the minimum

value of $|z|$ is the $\perp^r$ distance from the origin to the line segment c.

Thus on c, $|z| \geq 1/\sqrt{2}$.

So that $|z|^4 \geq \frac{1}{4} \Rightarrow \left|1/z^4\right| \leq 4$

By thm 6.2,

$$\left| \int_c \frac{dz}{z^4} \right| \leq 4\sqrt{2}.$$

pole:

Let a be an isolated singularity of $f(z)$ the point a is called a pole. If the principal part of $f(z)$ at $z=a$ has finite number of terms. If the principal part of $f(z)$ at $z=a$ is grn by

$$\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\dots+\frac{b_r}{(z-a)^r}$$ where $b_r \neq 0$ we say that a is a pole of order r for $f(z)$

If $\lim_{z \to a} f(z) = \infty$ the point a is said to be a pole of $f(z)$ and we set $f(a)=\infty$

Ex:

consider $f(z) = e^z/z$.

$$e^z/z = \frac{1}{z}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\dots\right)$$

$$= \frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+\dots$$

here the principal part of $f'(z)$ at $z=0$ has a single term $\frac{1}{z}$.

Hence $z=0$ is a simple pole of $f(z)$.

Essential singularity

Let a be an isolated singularity of $f(z)$. The point 'a' is called an essential singularity of $f(z)$ at $z=a$ is the principal part of $f(z)$ at $z=a$ has an infinite number of terms.

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

Eg: consider $f(z) = e^{1/z}$

Here $z=0$ is an isolated singularity for $f(z)$.

$$e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \cdots$$

$\therefore$ Here the principal part of $f(z)$ at $z=0$ has infinite number of terms.

Removable singularity

Let 'a' be an isolated singularity for $f(z)$. Then 'a' is called a removable singularity if the principal part of $f(z)$ at $z=a$ has no terms.

Ex:

Let $f(z) = \frac{\sin z}{z}$

0 is an isolated singular point for $f(z)$

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

Here the principle part of $f(z)$ at $z=0$ has no terms.

Hence $z=0$ is a removable singularity.

Unit - IV

* Calculus of residues

Unit - IV

Calculus of residues

Residues:

Let a be an isolated singularity for f(z). Then the residue of f(z) at a is defined to be coefficient of $\frac{1}{z-a}$ in the expansion of f(z) about a and is denoted by $\text{Res}\{f(z), a\}$.

V.Q Eg: Find residue?

consider $f(z) = \frac{e^z}{z^2}$

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$

$\therefore$ f(z) has a double pole at $z=0$

$\therefore \text{Res}\{f(z), 0\} = \text{coefficient of } \frac{1}{z} = 1$

Calculation of residues

i) If $z=a$ is a pole of order one of f(z) then $\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a}(z-a)f(z)$.

ii) If f(z) is of the form $\frac{h(z)}{g(z)}$ where $z=a$ is a simple pole $h(a)\neq 0$, $g(a)=0$ then $\underset{z\to a}{\text{Res}}\, f(z) = \frac{h(a)}{g'(a)}$, $g'(a)\neq 0$, $g(a)=0$.

iii) If f(z) has a pole of order m then, $\underset{z=a}{\text{Res}}\, f(z) = \frac{g^{m-1}(a)}{(m-1)!}$

1) Find the residue of the function $f(z) = \frac{z}{z^2+1}$

Soln:-

$$f(z) = \frac{z}{z^2+1}$$

$$z^2+1=0$$

$$z^2=-1$$

$$z=\pm i$$

$z = i, -i$ are poles

W.K.T

$$\underset{z=a}{\operatorname{Res}} f(z) = \lim_{z \to a} (z-a) f(z)$$

$$\underset{z=i}{\operatorname{Res}} f(z) = \lim_{z \to i} (z-i) \frac{z}{z^2+1}$$

$$= \lim_{z \to i} (z-i) \frac{z}{(z+i)(z-i)}$$

$$= \lim_{z \to i} \frac{z}{z+i}$$

$$= \frac{i}{i+i}$$

$$= \frac{i}{2i}$$

$$\underset{z=i}{\operatorname{Res}} f(z) = \frac{1}{2}$$

$$\underset{z=-i}{\operatorname{Res}} f(z) = \lim_{z \to -i} (z+i) \frac{z}{(z+i)(z-i)}$$

$$= \lim_{z \to -i} \frac{z}{z-i}$$

$$= \frac{-i}{-i-i}$$

$$= \frac{-i}{-2i}$$

$$\underset{z=-i}{\operatorname{Res}} f(z) = \frac{1}{2}$$

2) Find the residue of the function $f(z) = \tan z$ at $z = \pi/2$

Soln:-

$$f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{h(z)}{g(z)}$$

$z = \pi/2$ is a simple pole.

W.k.T

$$\operatorname*{Res}_{z=a} f(z) = \frac{h(a)}{g'(a)}$$

$$h(z) = \sin z$$

$$h(\pi/2) = \sin \pi/2$$

$$g(z) = \cos z$$

$$g'(z) = -\sin z \Rightarrow g'(\pi/2) = -\sin\pi/2$$

$$\operatorname*{Res}_{z=\pi/2} f(z) = \frac{h(\pi/2)}{g'(\pi/2)}$$

$$= \frac{\sin\pi/2}{-\sin\pi/2} = -1$$

3) $f(z) = \dfrac{1}{(z+1)^3}$

soln:-

$z=-1$ is a pole of order three

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{(3-1)!} \lim_{z\to -1} \left[(z+1)^3 \frac{1}{(z+1)^3}\right] \cdot \frac{d^2}{dz^2}$$

$$= \frac{1}{2!} \lim_{z\to -1} \frac{d^2}{dz^2}(1)$$

$$= 0$$

4) $f(z) = \dfrac{z+1}{z^2(z-2)}$

soln:-

$z=0$ (pole of order 2) and

$z=2$ (pole of order 1)

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z\to 0} \frac{d}{dz}\left[\frac{z^2(z+1)}{z^2(z-2)}\right]_{z=0}$$

$$= \frac{1}{1!} \lim_{z\to 0} \left[\frac{(z-2)(1)-(z+1)(1)}{(z-2)^2}\right]_{z=0}$$

$$= \lim_{z\to 0} \left[\frac{z-2-z-1}{(z-2)^2}\right]_{z=0}$$

$$= -3/4$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2) \frac{(z+1)}{z^2(z-2)} = 3/4$$

Thm: 4.1 Residue thm (or) Cauchy's residues thm.

Let $f(z)$ be analytic except for isolated singularities a_j in a region Ω then

$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j n(\gamma, a_j) \operatorname{Res}_{z=a_j} f(z)$ for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j

proof:-

case(i)

Let $f(z)$ has finitely many singularity points $a_1, a_2, \dots, a_n$

Let Ω' be the region obtained by omitting $a_1, a_2, \dots, a_n$.

For each a_j $\exists$ a $\delta_j > 0$ $\ni$: the corrected region $0 < |z-a_j| < \delta_j$ is contained in Ω'.

Draw a circle c_j about a_j of radius $< \varepsilon_j$ and

Let $P_j = \int_{c_j} f(z)dz$ — ① be the corresponding period of $f(z)$.

The particular function $\frac{1}{z-a_j}$ has the period $2\pi i$, we set $R_j = P_j/2\pi i$

The combination $f(z) = \frac{R_j}{z-a_j}$ has a vanishing period.

Let γ be a cycle in Ω' which is homologous to zero with respect to Ω then γ satisfy the homology.

$\gamma \sim \sum_j n(\gamma, a_j)\, c_j$ — ② with respect to Ω'

$$\int_\gamma f(z)\,dz = 1 \int_\gamma f(z)\,dz$$

$$= \sum_{j=1}^{n} n(\gamma, a_j) \int_{c_j} f(z)\,dz \quad \text{by } ②$$

$$= \sum_{j=1}^{n} n(\gamma, a_j) \cdot P_j \quad \text{by } ①$$

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\sum_{j=1}^{n} n(\gamma, a_j)\cdot P_j$$

$$= \frac{1}{2\pi i}\sum_{j=1}^{n} n(\gamma, a_j)\cdot R_j \cdot 2\pi i$$

$$= \sum_{j=1}^{n} n(\gamma, a_j)\, R_j$$

$$= \sum_{j=1}^{n} n(\gamma, a_j) \operatorname*{Res}_{z=a_j} f(z)$$

case(ii) :-

Suppose $f(z)$ has infinitely many singularity point a_j.

consider set $S = \{a \in \mathbb{C}, n(\gamma, a) = 0\}$ is open and contains all points outside of a large circle

It's component S^c if closed and bounded

$\therefore S^c$ is compact

and it cannot contain more that a finite number or isolated points a_j.

$\Rightarrow n(\gamma, a_j) \neq 0$ only for finite number of singular point.

By above case,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = n(\gamma, a_j) \operatorname*{Res}_{z=a_j} f(x).$$

1) Evaluate $\int_C \frac{2z^2+z}{z^2-1} dz$ where c is the circle

i) $c : |z-1| = 1$ ii) $c : |z| = 2$.

soln:-

$$\frac{2z^2+z}{z^2-1} = \frac{z(2z+1)}{(z+1)(z-1)}$$

Here the poles are $z = 1$, $z = -1$

W.K.T $R = \lim_{z\to a} (z-a) f(z)$

$$R_1 = \lim_{z\to 1} (z-1) \frac{2z^2+z}{(z+1)(z-1)}$$

$$= \lim_{z\to 1} \frac{2z^2+z}{(z+1)}$$

$$R_1 = \frac{2+1}{2} = \frac{3}{2}$$

where R_1 is the residue of the function at $z=1$.

since 1 is the only lying within the circle $|z-1| = 1$

By Residue thm,

$$\int_C f(z)dz = 2\pi i(R_1+R_2+\dots+R_n)$$

$$\int_C \frac{2z^2+z}{z^2-1} dz = 2\pi i(R_1)$$ ($z=-1$ is exterior, so do not find R_2)

$$= 2\pi i(3/2)$$

$$= 3\pi i$$

ii) Both the poles $z=1$, $z=-1$ are lie within the circle $|z| = 2$,

$$R_1 = \lim_{z\to 1} (z-1) \frac{2z^2+z}{(z+1)(z-1)}$$

$$= \lim_{z\to 1} \frac{2z^2+z}{z+1}$$

$$R_1 = 3/2$$

$$R_2 = \lim_{z \to -1} (z-(-1)) \frac{2z^2+z}{(z+1)(z-1)}$$

$$= \lim_{z \to -1} \frac{2z^2+z}{z-1}$$

$$R_2 = \frac{2(-1)^2+(-1)}{(-1)-1} = -\frac{1}{2}$$

By residue thm,

$$\int_c \frac{2z^2+z}{z^2-1} dz = 2\pi i (R_1 + R_2)$$

$$= 2\pi i \left(\frac{3}{2} - \frac{1}{2}\right)$$

$$= 2\pi i$$

2) Evaluate $\int_c \frac{dz}{z^3(z-1)}$ c is the circle $|z|=2$

soln:-

Given $f(z) = \frac{1}{z^3(z-1)}$

The poles are $z=0, z=1$

The residue at $z=0$ (here $z=0$ is a pole of order 3)

$$R_1 = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left(z^3 \cdot \frac{1}{z^3(z-1)}\right)$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz}\left(\frac{d}{dz}\left(\frac{1}{z-1}\right)\right)$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz}\left(\frac{-1}{(z-1)^2}\right)$$

$$= \frac{1}{2} \lim_{z \to 0} 2 \cdot \frac{1}{(z-1)^3}$$

$$= \frac{1}{2}(2 \cdot (-1))$$

$$R_1 = -1$$

The residue at $z=1$

$$R_2(z) = \lim_{z \to 1} (z-1) \frac{1}{z^3(z-1)}$$

$$= \lim_{z \to 1} \left(\frac{1}{z^3}\right)$$

$R_2(z) = 1$

$$\int \frac{dz}{z^3(z-1)} = 2\pi i (R_1 + R_2)$$
$$= 2\pi i (-1+1)$$
$$= 0.$$

Thm: 4.2 Argument principle.

If $f(z)$ is meromorphic in Ω with the zero a_j and the poles b_k then $\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz = \sum_j \eta(\gamma, a_j) - \sum_k \eta(\gamma, b_k)$ for every cycle γ which is homologous to zero in Ω and does not pass through any of the zero or poles.

proof:- Given $f(z) \neq 0$ in region Ω with zeros a_j, poles b_k with no other singularities in Ω.

Let $g(z)$ be analytic in Ω, γ be a homologous to zero in Ω not passing through a_j's and b_k's

To prove : $\frac{1}{2\pi i} \int_\gamma g(z) \frac{f'(z)}{f(z)} dz = \sum_j \eta(\gamma, a_j) g(a_j) - \sum_k \eta(\gamma, b_k) g(b_k)$ → (2)

If a_j is zero of order h for f then in a neighbourhood of a_j

$$f(z) = (z-a_j)^h f_1(z)$$

where $f_1(z)$ is analytic and $f_1(a_j) \neq 0$.

It follows from the continuity of $f_1(z)$ at a_j.

Then there is a neighbourhood of a_j in which $f_1(z) \neq 0$.

Thus in the neighbourhood $\frac{f'(z)}{f(z)} = \frac{h}{z-a_j} + \frac{f'(z)}{f(z)}$

Using Taylor's expansion for $g(z)$ at $z=a_j$

$$g(z) = g(a_j) + g(a_j)(z-a_j) + \cdots$$

and hence the Laurent expansion for $g(z)\frac{f'(z)}{f(z)}$ the coefficient of $1/z-a_j$.

which is nothing but

$\underset{z=a_j}{\text{Res}}\left[g(z)\frac{f'(z)}{f(z)}\right]$ is precisely $h - g(a_j)$

Thus $g(z) = \frac{f'(z)}{f(z)}$ has a pole at $z=a_j$ with residue $hg(a_j)$.

Similarly at a pole $z=b_k$ of order k for $f(z)$.

$g(z)\frac{f'(z)}{f(z)}$ also has a pole with residue $kg(b_k)$.

By using the residue thm,

we get $\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_j \eta(\gamma, a_j) g(a_j)$

$$- \sum_k \eta(\gamma, b_k)\, g(b_k)$$

where the terms are on the right hand side are to be repeated as many times as the order indicates

The above proof is called generalised of argument thm.

Take $g(z)=1$

we get the argument principle

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j \eta(\gamma, a_j) - \sum_k \eta(\gamma, b_k)$$

If c is a unit circle about the origin show that $\int_C \frac{e^{-z}}{z^2}\,dz = -2\pi i$

soln:- Given, $f(z) = \frac{e^{-z}}{z^2}$

The pole is zero '0', $z=0$

The residue at $z=0$ is

$$R = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{e^{-z}}{z^2}\right)$$

$$= \lim_{z\to 0} \frac{d}{dz} e^{-z}$$

$$= \lim_{z\to 0} e^{-z}(-1)$$

$$= \lim_{z\to 0} (-e^{z})$$

$$R = -1$$

$$\int_C \frac{e^{-z}}{z^2}\,dz = 2\pi i(R)$$

$$= -2\pi i$$

Thm: 4.3 Rouche's thm

Let γ be homologous to zero in Ω and such that $n(\gamma, z)$ is either 0 (or) 1 for any point z not on γ suppose that $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z)-g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the some number of zeroes enclosed by γ.

proof:- Let γ be homologous to zero in Ω

Let $f(z)$ and $g(z)$ be two analytic function in Ω and it satisfies the condition

$|f(z)-g(z)| < |f(z)|$ on γ

To prove:

$f(z)$ and $g(z)$ have the same number of zero's enclosed by γ.

From the hypothesis $f(z)$ and $g(z)$ do not vanish on γ.

From the given inequality

$$|f(z) - g(z)| < |f(z)|$$

$$\Rightarrow |g(z) - f(z)| < |f(z)|$$

$$\Rightarrow \left| \frac{g(z)}{f(z)} - 1 \right| < 1$$

Let $f(z) = \frac{g(z)}{f(z)}$

Let Γ be the image of γ under the map $f(z)$.

$$|f(z) - 1| < 1 \quad \forall\ z \text{ on } \gamma.$$

$\Rightarrow \Gamma$ lies in the open disk with centre 1 and radius 1 and does not wind around zero.

$$\eta(\Gamma, 0) = 0$$

ie) $\frac{1}{2\pi i} \int_{\Gamma} \frac{dw}{w} = 0$

ie) $\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = 0$

By argument principle (Number of zeroes of $f(z)$ inside γ) − (Number of poles of $f(z)$ inside γ) = 0

$\Rightarrow$ (No. of zero's of $g(z)$ inside γ) − (No of poles of $g(z)$ inside γ) = 0

ie) (No. of zero's of $g(z)$ inside γ) = (No of poles of $f(z)$)

$\therefore f(z)$ and $g(z)$ have the same number of zero's enclosed by γ.

chain and cycles:

i) Let $\gamma_1, \gamma_2, \dots, \gamma_n$ be arcs. A formal sum $\gamma_1 + \gamma_2 + \dots + \gamma_n = \gamma$ is called a chain.

ii) If $\gamma_1, \gamma_2, \dots, \gamma_n$ are all closed curves then γ is called a cycle.

Defn:

A cycle γ is said to be bound the region Ω iff $n(\gamma, a)$ is defined and equal to 1 for all points $a \in \Omega$ and either undefined or equal to zero for all points a not in Ω.

Type I

Define integrals of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$ where $F(\cos\theta, \sin\theta)$ is a real function of $\cos\theta$ and $\sin\theta$.

In such of problems we take the circle $|z|=1$ and $z = e^{i\theta}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}(z + 1/z),$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{1}{2i}(z - 1/z)$$

1) S.T $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ $(-1 < a < 1)$

Soln:- Let $z = e^{i\theta}$ be any point on the unit circle $|z|=1$ denoted by

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta$$

$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$1+a\sin\theta = 1+a\left\{\frac{1}{2i}\left(z-\frac{1}{z}\right)\right\}$$
$$= 1+a\left(\frac{z^2-1}{2iz}\right)$$
$$= \frac{2zi+az^2-a}{2zi}$$

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_{|z|=1} \frac{2zi}{2zi+az^2-a} \cdot \frac{dz}{iz}$$

$$= 2\int_C \frac{dz}{az^2+2iz-a}$$

$$= \frac{2}{a}\int_C \frac{dz}{z^2+\frac{2i}{a}z-1}$$

Let $f(z) = \dfrac{1}{z^2+\frac{2i}{a}z-1}$

$$\int_0 \frac{d\theta}{1+a\sin\theta} = \frac{2}{a}\int_C f(z)\,dz \quad \text{--- (1)}$$

The poles of $f(z)$ are given by

$$z^2+\frac{2i}{a}z-1=0$$

$$z = \frac{\left(-\frac{2i}{a}\right) \pm \sqrt{\left(\frac{2i}{a}\right)^2-4(1)(-1)}}{2(1)}$$

$$= \frac{-\frac{2i}{a} \pm \sqrt{\frac{4i^2}{a^2}+4}}{2}$$

$$= \frac{-\frac{2i}{a} \pm \frac{2i}{a}\sqrt{(1+a^2/i^2)}}{2}$$

$$= \frac{(-2i/a) \pm \frac{2i}{a}\sqrt{1-a^2}}{2}$$

$$= \frac{-i/a \pm i/a\sqrt{1-a^2}}{1}$$

$$= \frac{-i \pm i\sqrt{1-a^2}}{a}$$

$$z = \frac{-i+i\sqrt{1-a^2}}{a}, \quad \frac{-i-i\sqrt{1-a^2}}{a}$$

The poles of $f(z)$ are

$$\frac{-i+i\sqrt{1-a^2}}{a}, \quad \frac{-i-i\sqrt{1-a^2}}{a}$$

Let $\alpha = \frac{-i+i\sqrt{1-a^2}}{a}$ and $\beta = \frac{-i-i\sqrt{1-a^2}}{a}$

since $|\beta|>1$, $|\alpha|<1$

$z=\alpha$ is the only simple poles has inside the circle $|z|=1$.

$$f(z) = \frac{1}{(z-\alpha)(z-\beta)}$$

Residue of $f(z)$ at $z=\alpha$ is

$$R = \lim_{z\to\alpha} (z-\alpha) f(z)$$

$$= \lim_{z\to\alpha} (z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}$$

$$R = \lim_{z\to\alpha} \frac{1}{(z-\beta)}$$

$$R = \frac{1}{\alpha-\beta}$$

$$R = \frac{1}{\left(\frac{-i+i\sqrt{1-a^2}}{a}\right)-\left(\frac{-i-i\sqrt{1-a^2}}{a}\right)}$$

$$= \frac{a}{-i+i\sqrt{1-a^2}+i+i\sqrt{1-a^2}}$$

$$R = \frac{a}{2i\sqrt{1-a^2}}$$

By cauchy's Residue theorem,

$$\int_c f(z)dz = 2\pi i \text{ (sum of the residue)}$$

$$\int_c f(z)dz = 2\pi i \times R$$

$$= 2\pi i \times \frac{a}{2i\sqrt{1-a^2}}$$

$$= \frac{a\pi}{\sqrt{1-a^2}} \quad \text{—} ②$$

sub ② in ①

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2}{a}\int_C f(z)\,dz$$

$$= \frac{2}{a} \times \frac{a\pi}{\sqrt{1-a^2}}$$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$

2) Evaluate $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$

soln:-

$$\text{Let } I = \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{1+\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$= \int_{-\pi}^{\pi} \frac{2\,d\theta}{2+1-\cos 2\theta}$$

$$= \int_{-\pi}^{\pi} \frac{2\,d\theta}{3-\cos 2\theta}$$

$$= 2\int_{0}^{\pi} \frac{2\,d\theta}{3-\cos 2\theta}$$

$$= 4\int_{0}^{\pi} \frac{d\theta}{3-\cos 2\theta}$$

Let $2\theta = \phi$ when $\theta = 0 \Rightarrow \phi = 0$

$2\,d\theta = d\phi$ when $\theta = \pi \Rightarrow \phi = 2\pi$

$$\Rightarrow d\theta = \frac{d\phi}{2}$$

$$I = 4\int_0^{2\pi} \frac{d\phi/2}{3-\cos\phi}$$

$$I = \frac{4}{2}\int_0^{2\pi} \frac{d\phi}{3-\cos\phi}$$

Let $z = e^{i\phi}$ be any point on the unit circle $|z| = 1$ denoted by c

$$\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$3 - \cos\phi = 3 - \frac{1}{2}(z + 1/z)$$

$$= 3 - \left(\frac{z^2+1}{2z}\right)$$

$$= \frac{6z - z^2 - 1}{2z}$$

Now $z = e^{i\phi}$

$$dz = ie^{i\phi}\,d\phi$$

$$d\phi = \frac{dz}{ie^{i\phi}} = \frac{dz}{iz}$$

$$I = 2\int_c \frac{2z}{6z - z^2 - 1} \cdot \frac{dz}{iz}$$

$$= 2\int_c \frac{2}{-z^2+6z-1} \cdot \frac{dz}{i}$$

$$= \frac{4}{-i}\int_c \frac{dz}{z^2-6z+1} \quad \text{--- (1)}$$

$$= \frac{-4}{i}\int_c f(z)\,dz \quad \text{where } f(z) = \frac{1}{z^2-6z+1}$$

The poles of $f(z)$ are given by

$$z^2 - 6z + 1 = 0$$

$$z = \frac{6 \pm \sqrt{36-4}}{2}$$

$$= \frac{6 \pm \sqrt{32}}{2}$$

$$= \frac{6 \pm 4\sqrt{2}}{2}$$

$$z = 3 \pm 2\sqrt{2}$$

$\therefore$ The poles of $f(z)$ are $3+2\sqrt{2}$, $3-2\sqrt{2}$

Let $\alpha = 3+2\sqrt{2}$, $\beta = 3-2\sqrt{2}$

Since $|\alpha| > 1$ and $|\beta| < 1$

$z=\beta$ is the only simple pole lies inside c

Now, $f(z) = \dfrac{1}{(z-\alpha)(z-\beta)}$

Residue of $f(z)$ at $z=\beta$ is

$$R = \lim_{z\to\beta} (z-\beta) f(z)$$

$$= \lim_{z\to\beta} (z-\beta) \frac{1}{(z-\alpha)(z-\beta)}$$

$$R = \frac{1}{(\beta-\alpha)}$$

$$= \frac{1}{[(3-2\sqrt{2})-(3+2\sqrt{2})]}$$

$$= \frac{1}{3-2\sqrt{2}-3-2\sqrt{2}}$$

$$R = \frac{-1}{4\sqrt{2}}$$

By cauch Residue thm,

$$\int_c f(z)dz = 2\pi i \times \text{sum of the residues}$$

$$= 2\pi i \times \frac{1}{-4\sqrt{2}}$$

$$\int_c f(z)dz = \frac{-\pi i}{2\sqrt{2}} \quad \text{—— (2)}$$

sub (2) in (1)

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \frac{-4}{i}\int_c f(z)dz$$

$$= \frac{-4}{i} \times \frac{-\pi i}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}\sqrt{2}\pi}{\sqrt{2}}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi$$

3) Evaluate $\int_0^{\pi} \frac{d\theta}{17-8\cos\theta}$

soln:-

$$I = \frac{1}{2}\int_0^{\pi} \frac{d\theta}{17-8\cos\theta} \rightarrow (1)$$

Let $z = e^{i\theta}$ be any point on the unit circle $|z| = 1$.

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}\,d\theta$$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$17-8\cos\theta = 17-8\left(\frac{1}{2}\left(z+1/z\right)\right)$$

$$= 17-4\left(z+1/z\right)$$

$$= 17-\left(\frac{4z^2+4}{z}\right)$$

$$= \frac{17z-4z^2-4}{z}$$

$$\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{17-8\cos\theta} = \frac{1}{2}\int_{|z|=1}\frac{z}{17z-4z^2-4}\cdot\frac{dz}{iz}$$

$$= \frac{1}{2}\int_{|z|=1}\left(\frac{1}{i}\right)\left(\frac{dz}{17z-4z^2-4}\right)$$

$$= -\frac{1}{8i}\int\frac{dz}{z^2-\frac{17}{4}z+1}$$

Let $f(z) = \frac{1}{z^2-\frac{17}{4}z+1}$

The poles of $f(z)$ are given by

$$z^2-\left(\frac{17}{4}\right)z+1$$

$$z = \frac{+(17/4)\pm\sqrt{(17/4)^2-4}}{2}$$

$$= \frac{+17/4\pm\sqrt{289/16-4}}{2}$$

$$= \frac{+(17/4) \pm \sqrt{225/16}}{2}$$

$$= \frac{17/4 \pm 15/4}{2}$$

$$= \frac{17 \pm 15}{8}$$

$$z = \frac{17+15}{8}, \quad \frac{17-15}{8}$$

$$z = 4, \ 1/4$$

Since $|\alpha| > 1$, $|\beta| < 1$

$z = \beta$ is the only simple pole lies inside c.

Now, $f(z) = \dfrac{1}{(z-\alpha)(z-\beta)}$

Residue of $f(z)$ at $z = \beta$ is,

$$R = \lim_{z \to \beta} (z-\beta) \frac{1}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z \to \beta} \frac{1}{(z-\alpha)}$$

$$= \frac{1}{\left(\frac{1}{4} - 4\right)}$$

$$= -4/15$$

By cauchy's residue theorem,

$$\int_c f(z)\,dz = 2\pi i \times \text{sum of the residues}$$

$$= 2\pi i \times -4/15$$

$$= \frac{-8\pi i}{15} \quad \text{—②}$$

sub ② in ①

$$\frac{1}{2} \int_0^{2\pi} \frac{d\theta}{17 - 8\cos\theta} = \frac{-1}{8i} \int_c f(z)\,dz$$

$$\frac{1}{2} \int_0^{2\pi} \frac{d\theta}{17 - 8\cos\theta} = -\frac{1}{8i} \times \frac{-8\pi i}{15}$$

$$= \frac{\pi}{15}$$

$$\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta} = \frac{2\pi}{15}.$$

Type - II

- Integrals of the form $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}$ where $p(x)$, $q(x)$ are polynomials and $\deg q(x) > 1 + \deg p(x)$ and number of poles lies on the real axis.

1) Evaluate $\int_0^{\infty} \frac{dx}{x^2+1}$

soln:-

Let $f(z) = \frac{1}{z^2+1}$

poles of $f(z)$ are given by

$$z^2+1=0$$

$$z^2=-1$$

$$z=\pm i$$

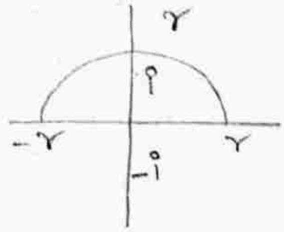
The poles are $z = i, -i$.

choose the contour c consists of the interval $[-r, r]$ on the real axis and the semicircle c with centre c and radius r that lies in the upper half plane.

$$\int_{-r}^{r} f(x)\,dx + \int_{c_1} f(z)\,dz = \int_c f(z)\,dz \quad -①$$

The pole $z=i$ lies inside c.

The residue of $f(z)$ at $z=i$



$$R = \lim_{z\to i} (z-i)\, f(z)$$

$$= \lim_{z\to i} (z-i) \frac{1}{(z-i)(z+i)}$$

$$= \lim_{z\to i} \frac{1}{(z+i)}$$

$$R = \frac{1}{2i}$$

By cauchy residue theorem

$$\int_C f(z)\,dz = 2\pi i \times R$$
$$= 2\pi i \times \frac{1}{2i} = \pi$$

From ①,

$$\int_C f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz$$

$$\int_{C_1} f(z)\,dz = 0 \text{ as } r \to \infty$$

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\pi = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

$$= 2\int_0^{\infty} \frac{1}{x^2+1}\,dx$$

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \pi/2.$$

2) Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$

soln:- Let $f(z) = \frac{1}{(z^2+a^2)^2}$

The poles of $f(z)$ are

$$(z^2+a^2)^2 = 0 \Rightarrow z^2+a^2 = 0$$
$$z^2 = -a^2$$
$$z = \pm ai$$

$z = ia, -ia$ are double poles

Choose the counter c consists of the interval $[-r, r]$ on the real axis and the semicircle C_1 with centre 0 and the radius r that lies in the upper half plane.

$$\int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz = \int_C f(z)\,dz \rightarrow ①$$

The pole $z = ai$ lies inside c $z = ai$ is a double pole Res of $f(z)$ at $z = ai$ is

$$R = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

$$R = \underset{z=ai}{\text{Res}}\, f(z) = \frac{1}{1!} \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{(z^2+a^2)^2}\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{((z-ai)(z+ai))^2}\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]$$

$$= \lim_{z\to ai}\left[\frac{-2}{(z+ai)^3}\right] = \frac{-2}{(ai+ai)^3}$$

$$R = \frac{-2}{(2ai)^3} = \frac{-2}{8i^3a^3} = \frac{-1}{4ia^3}$$

By cauchys residue thm,

$$\int_C f(z)\,dz = 2\pi i R$$

$$= 2\pi i \times \frac{1}{4ia^3}$$

$$= \frac{\pi}{2a^3} \quad \text{—} ②$$

① becomes

$$\int_C f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz$$

as $r\to\infty$ $\int_{C_1} f(z)dz = 0$

$$\int_C f(z)\,dz = \int_{-r}^{r} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

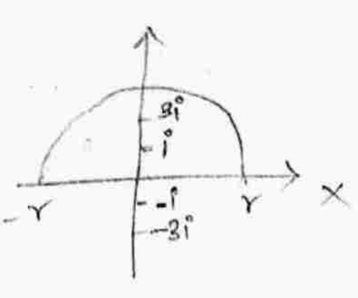
$$2\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

3) Evaluate: $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx.$

soln:-

Let $f(z) = \dfrac{z^2-z+2}{z^4+10z^2+9}$



The poles of $f(z)$ are

z^4+10z^2+9

$\Rightarrow (z^2+9)(z^2+1)=0$

$\Rightarrow z = \pm 3i, \quad z = \pm i$

$z = 3i, -3i, i, -i$ are simple poles.

Choose the counter consists of the interval $[-r, r]$ on the real axis and the semicircle with centre 0 and radius r that lies in the upper half plane.

$$\int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz = \int_{C} f(z)dz \quad \text{—} \; (1)$$

The poles of $f(z)$ lying within C are $i, 3i$ and both of them are simple poles.

Res of $f(z)$ at $z=i$ is

$$R_1 = \frac{h(i)}{k(i)}$$

$k(z) = z^4+10z^2+9$

$k'(z) = 4z^3+20z$

$k'(i) = 4i^3+20i$

$= 16i.$

$h(z) = z^2 - z + 2$

$h(i) = i^2 - i + 2 = 1 - i$

$$R_1 = \frac{1-i}{16i}$$

The Residue of $f(z)$ at $z=3i$

$$R_2 = \frac{h(3i)}{k'(3i)}$$

$h(z) = z^2 - z + 2$

$h(3i) = (3i)^2 - 3i + 2 = 9i^2 - 3i + 2 = -3i - 7$

$k(z) = z^4 + 10z^2 + 9$

$k'(z) = 4z^3 + 20z$

$k'(3i) = 4(3i)^3 + 20(3i) = -108i + 60i = -48i$

$$R_2 = \frac{-3i-7}{-48i} = \frac{3i+7}{48i}$$

By cauchy's residue thm,

$\int_C f(z)dz = 2\pi i \times$ sum of the residue

$= 2\pi i [R_1 + R_2]$

$$= 2\pi i \left[\frac{1-i}{16i} + \frac{7+3i}{48i} \right]$$

$$= 2\pi i \left[\frac{3-3i+7+3i}{48i} \right]$$

$$= 2\pi i \times \frac{10}{48i}$$

$$= \frac{5\pi}{12}$$

From ①

$$\int_{-r}^{r} \frac{x^2-x+2}{x^4+10x^2+9}\, dx + \int_{C_1} f(z)\, dz = \frac{5\pi}{12}$$

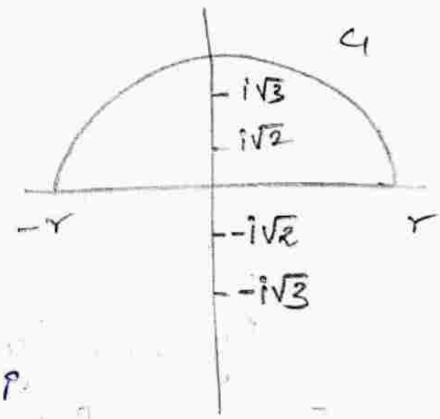
as $r \to \infty$, $\int_{C_1} f(z)dz = 0$

$$\Rightarrow \int_{-r}^{r} \frac{x^2-x+2}{x^4+10x^2+9}\, dx = \frac{5\pi}{12}$$

4) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}\, dx$.

soln:-

Let $f(z) = \frac{z^2}{z^4+5z^2+6}$



The poles of $f(z)$ are

$$z^4+5z^2+6 = 0$$

$$(z^2+3)(z^2+2) = 0$$

$$z = \pm\sqrt{3}\,i, \quad z = \pm\sqrt{2}\, i$$

$z = \sqrt{3}\,i, -i\sqrt{3}, i\sqrt{2}, -i\sqrt{2}$ are simple poles of $f(z)$.

choose the countour consists of the interval $[r, r]$ on the real axis and the semi circle with centre 0 and radius r that lies in the upper half plane.

$$\int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz = \int_{C} f(z)dz \quad \text{—①}$$

The poles of $f(z)$ lying within c are $i\sqrt{3}$ and $i\sqrt{2}$ and both of them are simple poles.

Res of $f(z)$ at $z = i\sqrt{3}$ is

$$R_1 = \frac{h(i\sqrt{3})}{k'(i\sqrt{3})}$$

$h(z) = z^2$

$h(i\sqrt{3}) = -3$

$K(z) = z^4 + 5z^2 + 6$

$K'(z) = 4z^3 + 10z$

$K'(i\sqrt{3}) = 4(i\sqrt{3})^3 + 10(i\sqrt{3})$

$= -12\sqrt{3}i + 10\sqrt{3}i$

$= -2\sqrt{3}i$.

$$R_1 = \frac{-3}{-2\sqrt{3}i} = \frac{\sqrt{3}}{2i}$$

Res of $f(z)$ at $z = i\sqrt{2}$

$$R_2 = \frac{h(i\sqrt{2})}{K'(i\sqrt{2})}$$

$h(z) = z^2$

$h(i\sqrt{2}) = -2$

$K'(z) = 4z^3 + 10z$

$K'(i\sqrt{2}) = 4(i\sqrt{2})^3 + 10(i\sqrt{2})$

$= -8\sqrt{2}i + 10\sqrt{2}i$

$= 2\sqrt{2}i$

$$R_2 = \frac{-2}{2\sqrt{2}i} = \frac{-1}{\sqrt{2}i}$$

By cauchy's Residue thm,

$\int_C f(z)dz = 2\pi i \times$ Sum of the residue

$= 2\pi i(R_1 + R_2)$

$= 2\pi i\left(\frac{\sqrt{3}}{2i} - \frac{1}{\sqrt{2}i}\right)$

$= 2\pi i\left(\frac{\sqrt{3} - \sqrt{2}}{2i}\right)$

$= \pi(\sqrt{3} - \sqrt{2})$

From ①

$$\int_{-r}^{r} \frac{x^2}{x^4+5x^2+6}\,dx + \int_{C_1} f(z)dz = \pi(\sqrt{3} - \sqrt{2})$$

as $r \to \infty$ $\int_{C_1} f(z)dz = 0$

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}\, dx = \pi(\sqrt{3}-\sqrt{2})$$

$$2\int_{0}^{\infty} \frac{x^2}{x^4+5x^2+6}\, dx = \pi(\sqrt{3}-\sqrt{2})$$

$$\Rightarrow \int_{0}^{\infty} \frac{x^2}{x^4+5x^2+6}\, dx = \frac{\pi}{2}(\sqrt{3}-\sqrt{2}).$$

Type III:

The integral of the form

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin x\, dx \text{ (or) } \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos x\, dx$$

where $p(x)$ and $q(x)$ are real polynomial and $q(x)$ has no zeroes on the real axis

1) Evaluate $\int_{0}^{\infty} \frac{\cos mx}{a^2+x^2}\, dx$.

soln:-

consider $f(z) = \frac{e^{imz}}{a^2+z^2}$

choose the contour c consisting of the interval $[-r, r]$ on the real axis and the semi-circle c_1 with centre 0 and radius r that lies in that upper half plane.

$$\int_{-r}^{r} f(x)\,dx + \int_{c_1} f(z)\,dz = \int_{c} f(z)\,dz \quad \text{— (1)}$$

The poles of $f(z)$ are

$$a^2+z^2=0$$

$$z^2=-a^2 \Rightarrow z=\pm ai$$

$z=ai$ is the only simple poles inside c.

Residue of $f(z)$ at $z=ai$

$$R = \underset{z=ai}{\text{Res}}\, f(z) = \lim_{z\to ai} (z-ai) f(z)$$

$$= \lim_{z\to ai} (z-ai) \times \frac{e^{imz}}{a^2+z^2}$$

$$= \lim_{z\to ai} (z-ai) \frac{e^{imz}}{(z+ia)(z-ia)}$$

$$= \lim_{z \to ai} \frac{e^{imz}}{(z+ia)}$$

$$= \frac{e^{imia}}{ai+ai}$$

$$= \frac{e^{-ma}}{2ia}$$

By cauchy residue thm,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of the residues}$$

$$= 2\pi i \times R$$

$$= 2\pi i \times \frac{e^{-ma}}{2ia} = \frac{\pi e^{-ma}}{a} \quad \text{—②}$$

Using ② in ①

$$\frac{\pi}{a} e^{-ma} = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz$$

as $r \to \infty$ $\int_{C_1} f(z)\,dz = 0$

$$\frac{\pi}{a} e^{-ma} = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\frac{\pi}{a} e^{-ma} = \int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{a^2+x^2}\,dx = \frac{\pi}{a} e^{-ma}$$

Equating the real part

$$\int_{-\infty}^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi}{a} e^{-ma}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi}{a} e^{-ma}$$

$$\int_{0}^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi}{2a} e^{-ma}$$

2) Evaluate $\int_{-\infty}^{\infty} \frac{\cos 3x}{1+x^2}\,dx$.

Soln:- consider $f(z) = \frac{e^{imz}}{a^2+z^2}$

Choose the contour C consisting of the

interval $[-r, r]$ on the real axis and the semicircle C_1 with centre 0 and radius r that lies in that upper half plane

$$\int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz = \int_C f(z)\,dz \quad \text{— (1)}$$

The poles of $f(z)$ are

$$1+z^2=0$$
$$z = \pm i$$

$z=i$ is the only simple pole inside c

Residue of $f(z)$ at $z=i$

$$R = \operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} (z-i) f(z)$$
$$= \lim_{z\to i} (z-i) \frac{e^{i3z}}{1+z^2}$$
$$= \lim_{z\to i} (z-i) \frac{e^{i3z}}{(z+i)(z-i)}$$
$$= \lim_{z\to i} \frac{e^{i3z}}{z+i}$$
$$= \frac{e^{i3i}}{i+i}$$
$$= \frac{e^{-3}}{2i}$$

By cauchy residue thm,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of the residue}$$
$$= 2\pi i \times R$$
$$= 2\pi i \times \frac{e^{-3}}{2i}$$
$$= \pi e^{-3} \quad \text{— (2)}$$

Using (2) in (1)

$$\frac{\pi}{a} e^{-ma} = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz$$

as $r \to \infty$, $\int_{C_1} f(z)\,dz = 0$

$$\pi e^{-3} = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{i3x}}{1+x^2}\, dx = \pi e^{-3}$$

$$\int_{-\infty}^{\infty} \frac{\cos 3x + i\sin 3x}{1+x^2}\, dx = \pi e^{-3}$$

Equating the real part

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{1+x^2}\, dx = \pi e^{-3}$$

U.Q 1) How many roots of the eqn $z^4 - 6z + 3$ have their modulus between 1 and 2.

soln:-

Given that

The eqn $z^4 - 6z + 3 = 0$

$1 \le z \le 2$

Here $z = 1$ and $z = 2$

$|f(z) + g(z)| = z^4 - 6z + 3$

$z = 1$ $\qquad$ $z = 2$

$f(z) = -6z$ $\qquad$ $f(z) = 6z$

$g(z) = z^4 + 3$ $\qquad$ $g(z) = -6z + 3$

By Rouchy thm

$|f(z)| > |g(z)|$

$z = 1$

$|f(z)| = |-6z| = 6 \cdot 1 = 6 \Rightarrow 1$

$|g(z)| = z^4 + 3 = 4 \Rightarrow 4$

The number of zero's

$= 4 - 1 = 3$

$z = 2$

$|f(z)| = |z^4| = |2|^4 = 16 \Rightarrow 4$

$|g(z)| = |-6z + 3| = 15 \Rightarrow 1$

The number of zero's

$4 - 1 = 3$

$\therefore$ The number of zero's $= 3$

U.Q 2) Evaluate $\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx$, by the method of residues

Soln:-

Given that,

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx$$

$$f(z) = \frac{z.e^{iz}}{z^2+a^2}\,dz$$

poles $z^2+a^2=0$

$$z^2 = -a^2$$

$$z = \pm ia$$

$$R = \operatorname*{Res}_{z=ai} f(z) = \lim_{z\to ai} (z-ai)\frac{ze^{iz}}{(z+ai)(z-ai)}$$

$$= \frac{aie^{i(ai)}}{ai+ai}$$

$$= \frac{aie^{-a}}{2ai}$$

$$R = \frac{e^{-a}}{2}$$

Residues $\Rightarrow \int_C f(z)\,dz = 2\pi i(R)$

$$= 2\pi i \times \frac{e^{-a}}{2}$$

$$\int_C f(z)\,dz = \pi i e^{-a}$$

Here, $\int_{-\infty}^{\infty} f(x)\,dx = \int_C f(z)\,dz$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x[\cos x + i\sin x]}{x^2+a^2}\,dx = \pi i e^{-a}$$

$$2\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx = \pi e^{-a}$$

Hence $\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-a}$.

3) Evaluate find the residue of $f(z)=\frac{1}{2z+3}$

soln:-

poles $f(z)=\frac{1}{2z+3}$

$2z+3=0 \Rightarrow 2z=-3$

$z=-3/2$

Residue at $z=-3/2$

$$= \lim_{z\to -3/2} (z+3/2)\frac{1}{2(z+3/2)}$$

$$= \lim_{z\to -3/2} \frac{1}{2}$$

4) Find the residue $\frac{z}{z^2-1}=\frac{z}{(z+1)(z-1)}$

soln:-

poles $z^2-1=0$

$z^2=1=\pm 1$

Residue at $z=-1 \Rightarrow \lim_{z\to -1}(z+1)\cdot\frac{z}{(z+1)(z-1)}=\frac{1}{2}$

Residue at $z=1 \Rightarrow \lim_{z\to 1}(z-1)\frac{z}{(z+1)(z-1)}=\frac{1}{2}$

5) Find the residue of $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ at $z=2$

soln:-

$f(z)=\frac{z^2}{(z-1)^2(z+2)}$ at $z=-2$

Residue at $z=-2$; $R_1=\lim_{z\to -2}(z+2)\frac{z^2}{(z-1)^2(z+2)}$

$$R_1=\frac{(-2)^2}{(-2-1)^2}=\frac{4}{9}$$

Residue at $z=1$, $R_2=\lim_{z\to 1}\frac{d}{dz}(z-1)^2\frac{z^2}{(z-1)(z+2)}$

$$=\lim_{z\to 1}\frac{(z+2)(2z-z^2(1))}{(z+2)^2}$$

$$=\lim_{z\to 1}\left(\frac{2z^2+4z-z^2}{(z+2)^2}\right)$$

$$= \lim_{z \to 1} \frac{z^2+4z}{(z+2)^2}$$

$$R_2 = \frac{1+4}{3^2} = \frac{5}{9}$$

Total residue $= R_1 + R_2 = \left(\frac{4}{9} + \frac{5}{9}\right) 2\pi i$

$$= \frac{9}{9} 2\pi i$$

Total residue $= 2\pi i$.

Unit - $\overline{V}$

* power series expansion
* Canonical products
* Jensen's formula

Unit - $\overline{V}$

Weistrass theorem:

10m

Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω, uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω. Moreover $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω

proof:

The analyticity of $f(z)$ follows most easily by use of Morera's theorem, "if $f(z)$ is defined and continuous in Ω and if $\int_\gamma f(z)dz=0$ $\forall$ closed curve in Ω then $f(z)$ is analytic in Ω."

Let $|z-a| < r$ be a closed disk contained in Ω.

The assumption implies that the disk lies in Ω_n

If γ is any closed curve in $|z-a|<r$ we have, $\int_\gamma f_n(z)\,dz = 0$ $\forall$ $n \geq n_0$.

By cauchy's thm,

"If $f(z)$ is analytic in Ω the $\int_\gamma f(z)dz = 0$ for every curve γ"

The uniform converges on γ.

$\int_\gamma f(z)dz = \lim\limits_{n\to\infty} \int_\gamma f(z)dz = 0$ and by Morera's thm

It follows that $f(z)$ is analytic in $|z-a|<r$

$\therefore f(z)$ is analytic in Ω.

An alternative and more-explicit proof is based on the integral formula.

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\xi)}{\xi - z} d\xi$$ where c is the circle $|\xi - a| = r$, $|z-a| < r$ and $n \to \infty$

By uniform converges thm

$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi$ and this formula and show that $f(z)$ is analytic in a disc.

From the formula $f_n'(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\xi)}{(\xi - z)^2} d\xi$

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^2} d\xi = f'(z)$$

$\therefore f'(z)$ is analytic

claim:

$\{f_n(z)\}$ converges to $f'(z)$ uniformly on compact subset of Ω

Let $|z-a| < \delta$ and $\delta < r$ be a compact subset of Ω

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_C \frac{f_n(\xi)}{(\xi - z)^2} d\xi - \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^2} d\xi \right|$$

$$= \left| \frac{1}{2\pi i} \int_C \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2} d\xi \right|$$

$$\leq \frac{1}{2\pi} \int_C \frac{|f_n(\xi) - f(\xi)|}{|(\xi - z)^2|} |d\xi| \quad \text{---} (1)$$

$$|\xi - z| = |\xi - a + a - z| = |\xi - a - (z-a)|$$

$$|\xi - z| \geq r - \delta$$

$$\frac{1}{|\xi - z|} \leq \frac{1}{r - \delta} \Rightarrow \frac{1}{|\xi - z|^2} \leq \frac{1}{(r-\delta)^2}$$

since $f_n(z) \to f(z)$ uniformly given $\varepsilon > 0$

$|f_n(z) - f(z)| < \varepsilon \quad \forall\ n \geq n_0\ \forall\ z$

① becomes

$$|f_n'(z) - f'(z)| < \frac{1}{2\pi} \int_C \frac{\varepsilon}{(r-\delta)^2} |d\zeta| \quad \forall\ n \geq n_0$$

$$< \frac{\varepsilon}{2\pi(r-\delta)^2} \int_C |d\zeta|$$

$$< \frac{\varepsilon}{2\pi(r-\delta)^2}\ 2\pi$$

$$|f_n'(z) - f'(z)| < \varepsilon/(r-\delta)^2.$$

Since ε is arbitrary

$f_n'(z) \to f'(z) \to 0$ uniformly $\forall\ z$

$0 \leq |z-a| < \delta < r$

Since every compact set can be covered by a finite number of closed disk $0 \leq |z-a| < \delta < r$

The convergence is uniform on every compact subset of Ω

Hence proved.

Hurwitz theorem:

If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.

proof:-

Suppose that $f(z)$ is not identically zero.

W.K.T The zeroes of $f(z)$ are isolated.

For any point $z_0 \in \Omega$ of $r > 0 \ni : f(z)$ is defined and $\neq 0$ for $0 < |z - z_0| \leq r$

$|f(z)|$ has a positive minimum on the circle be $|z - z_0| = r$, we denote by c.

By hypothesis, $f_n(x) \to f(z)$ uniformly on a compact subset of Ω.

$f_n(z) \to f(z)$ uniformly on the circle c.

$\Rightarrow \frac{1}{f_n(z)} \to \frac{1}{f(x)}$ uniformly on c.

By Weistrass thm,

$f_n'(x) \to f'(z)$ uniformly on every compact subset of Ω:

$\Rightarrow f_n'(x) \to f'(z)$ uniformly on c.

$$\frac{f_n'(x)}{f_n(z)} \to \frac{f'(z)}{f(z)}$$

ie) $$\lim_{n\to\infty} \frac{1}{2\pi i} \int_c \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} dz \quad -①$$

But the L.H.S are zero ~~for~~ by thus given the number of roots of eqn $f_n(z)=0$.

$$① \Rightarrow \frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} dz = 0$$ in which $f(z_0) \neq 0$

z_0 is arbitrary

$\therefore f(z)$ never equal to zero in Ω.

Hence proved.

Taylor's series

If $f(z)$ is analytic in the region Ω, containing z_0, then the representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \dots$$

is valid in the largest open disk of center z_0 contained in Ω.

proof:- Let $r>0$ be such that the disk $|z-z_0|<r$ is contained in Ω

Let $0<r_1<r$.

Let c_1 be the circle $|z-z_0|=r_1$.

By cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{c_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta \quad \text{— ①}$$

Also by theorem on higher derivatives, we have

$$f^{n}(z) = \frac{n!}{2\pi i} \int_{c_1} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad \text{— ②}$$

Now,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$

$$= \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \dots + \left(\frac{z - z_0}{\zeta - z_0}\right)^{n-1} + \frac{\left(\frac{z - z_0}{\zeta - z_0}\right)^n}{\left[1 - \left(\frac{z - z_0}{\zeta - z_0}\right)\right]}\right]$$

[using $\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}$]

$$= \frac{1}{(\zeta - z_0)} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \dots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \frac{(z - z_0)^n}{(\zeta - z_0)^n (\zeta - z)}$$

Now multiplying throughout by $\frac{f(\zeta)}{2\pi i}$ integrating over c_1 and using ① & ② we get

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n \quad \text{— ③}$$

where $R_n = \frac{(z - z_0)^n}{2\pi i} \int_{c} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta$

Here ζ lies on c_1 and z lies in the interior of c_1

So that $|\zeta - z_0| = r_1$ and $|z - z_0| < r_1$

$$|\zeta-z| = |(\zeta-z_0)-(z-z_0)|$$
$$\geq |\zeta-z_0| - |z-z_0|$$
$$= r_1 - |z-z_0|$$
$$\frac{1}{|\zeta-z|} \leq \frac{1}{r_1-|z-z_0|}.$$

Let M denote the maximum value of $|f(z)|$ on C_1

Then $$|R_n| \leq \frac{|z-z_0|^n}{2\pi} \frac{M(2\pi r_1)}{(r_1-|z-z_0|).r_1^n}$$
$$= \frac{M|z-z_0|}{(r_1-|z-z_0|)}\left(\frac{|z-z_0|}{r_1}\right)^{n-1}$$

Also, $\left|\frac{z-z_0}{r_1}\right| < 1$. Hence $\lim_{n\to\infty} R_n = 0$

Taking limit as $n\to\infty$ in ③ we get

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^2 + \dots$$

Hence proved.

pblms:

Expand $\cos z$ into a ~~...~~ Taylor's series about the point $z=\pi/2$ and determine the radius of convergence.

soln:

Let $f(z) = \cos z$.

The Taylor's series

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$$

about $z=\pi/2$

$$f(z) = f(\pi/2) + \frac{(z-\pi/2)}{1!}f'(\pi/2) + \frac{(z-\pi/2)^2}{2!}f''(\pi/2) + \frac{(z-\pi/2)^3}{3!}f'''(\pi/2) + \dots$$

$f(z) = \cos z \Rightarrow f(\pi/2) = \cos \pi/2 = 0$

$f'(z) = -\sin z \Rightarrow f'(\pi/2) = -\sin \pi/2 = -1$

$f''(z) = -\cos z \Rightarrow f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(z) = \sin z \Rightarrow f'''(\pi/2) = \sin \pi/2 = 1$

$\vdots \qquad\qquad \vdots$

The Taylor's series for $\cos z$ about $z = \pi/2$ is

$$\cos z = \frac{-(z-\pi/2)}{1!} + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$

The expansion is valid throughout the complex plane.

2) Expand $\sin z$ into a Taylor's series about the point $z = \pi/4$ and determine the region of convergence.

soln:- Let $f(z) = \sin z$

The Taylor's series,

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

about $z_0 = \pi/4$

$$f(z) = f(\pi/4) + \frac{(z-\pi/4)}{1!} f'(\pi/4) + \cdots$$

$f(z) = \sin z \Rightarrow f(\pi/4) = \sin \pi/4 = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z \Rightarrow f'(\pi/4) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z \Rightarrow f''(\pi/4) = -\sin \pi/4 = \frac{-1}{\sqrt{2}}$

$f'''(z) = -\cos z \Rightarrow f'''(\pi/4) = -\cos \pi/4 = \frac{-1}{\sqrt{2}}$

The Taylor's series for $\sin z$ about $z = \pi/4$ is

$$\sin z = \frac{1}{\sqrt{2}} + \frac{(z-\pi/4)}{1!}\left(\frac{1}{\sqrt{2}}\right) - \frac{(z-\pi/4)^2}{2!}\left(\frac{1}{\sqrt{2}}\right) - \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{(z-\pi/4)}{1!} - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right]$$

The expansion is valid for entire complex plane.

3) $f(z) = \frac{z-1}{z+1}$ as a Taylor's series i) about the point $z=0$ ii) about the point $z=1$. Determine the region of convergence in each case.

soln:-

i) $f(z) = \frac{z-1}{z+1}$

$$= (z-1)(z+1)^{-1}$$

$$= (z-1)(1-z+z^2-z^3+\dots) \text{ if } |z|<1$$

$$= (z-z^2+z^3-\dots) - (1-z+z^2-z^3+\dots)$$

$$= -1+2z-2z^2+2z^3-\dots$$

The region of convergence is $|z|<1$

ii) $f(z) = \frac{z-1}{z+1}$

$$= \frac{z-1}{2+z-1}$$

$$= \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$$

$$= \left(\frac{z-1}{2}\right)\left(1+\left(\frac{z-1}{2}\right)\right)^{-1}$$

$$= \left(\frac{z-1}{2}\right)\left[1-\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{3}\right)^3+\dots\right]$$

if $\left|\frac{z-1}{2}\right|<1$

$$= \frac{z-1}{2} - \left(\frac{z-1}{2}\right)^2 + \left(\frac{z-1}{3}\right)^2 - \dots$$

The region of convergence is given by $\left|\frac{z-1}{2}\right|<1$ which is same as the circular.

4) Expand ze^{2z} in Taylor series about $z=-1$ and determine the region of convergence.

soln:-

Let $f(z) = ze^{2z}$

$$= ze^{2(z+1)}e^{-2}$$

$$= \frac{1}{e^2}\left((z+1)e^{2(z+1)} - e^{2(z+1)}\right).$$

$$= \frac{1}{e^2}\left[(z+1)\left\{1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \dots\right\} - \left\{1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \dots\right\}\right]$$

$$= \frac{1}{e^2}\left[\left\{(z+1) + \frac{2(z+1)^2}{1!} + 2^2\frac{(z+1)^3}{2!} + \dots\right\} - \left\{1 + \frac{2(z+1)}{1!} + \frac{2^2(z+1)^2}{2!} + \dots\right\}\right.$$

$$= \frac{1}{e^2}\left(-1 + \left(1 - \frac{2}{1!}\right)(z+1) + \left(\frac{2}{1!} - \frac{2^2}{2!}\right)(z+1)^2 + \left(\frac{2^2}{2!} - \frac{2^3}{3!}\right)(z+1)^3 + \dots\right.$$

The expansion is valid throughout the complex plane.

5) Find the Taylor's series of $\frac{z^2-1}{(z+2)(z+3)}$ i) $|z|<2$ ii) $2<|z|<3$ iii) $|z|<3$.

Soln:

Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

$$\begin{array}{r|l} & 1 \\ z^2+5z+6 & z^2+0z-1 \\ & z^2+5z-6 \\ \hline & -5z-7 \end{array}$$

$$f(z) = 1 - \frac{(5z+7)}{z^2+5z+6}$$

$$\frac{5z+7}{z^2+5z+6} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$5z+7 = A(z+3) + B(z+2)$$

$z=-3$,

$$-15+7 = 0 + B(-1)$$

$$-B = -8$$

$$B = 8$$

$z=-2$,

$$-10+7 = A(1)+0$$

$$A = -3$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$\cdot\ z+2=0 \Rightarrow z=-2$

$z+3=0 \Rightarrow z=-3$

singularities,

i) $|z|<2 \Rightarrow \frac{|z|}{2}<1$

$$f(z) = 1+\frac{3}{2}(1+z/2)^{-1} - 8/3\,(1+z/3)^{-1} + \ldots$$

$$= 1+\frac{3}{2}\left(1-z/2+z^2/2^2-\ldots\right) - \frac{8}{3}\left(1-z/3+z^2/3^2+\ldots\right)$$

$|z|<2$

Its necessity <3.

Hence the expansion is valid $|z|<2$

ii) $2<|z|<3$

$2<|z|$; $|z|<3$

$\frac{2}{|z|}<1$; $\frac{|z|}{3}<1$

$\left|\frac{2}{z}\right|<1$; $\left|\frac{z}{3}\right|<1$

$$f(z) = 1+\frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$$

$$= 1+\frac{3}{z}(1+2/z)^{-1} - 8/3\,(1+z/3)^{-1}$$

$$= 1+\frac{3}{z}\left[1-2/z+(2/z)^2-(2/z)^3+\ldots\right]$$

$$- 8/3\left[1-z/3+(z/3)^2-(z/3)^3+\ldots\right]$$

iii) $|z|>3$

$\left|\frac{1}{z}\right|<\frac{1}{3}$

$\left|\frac{3}{z}\right|<1$ $\quad \frac{2}{z}<1$

$$f(z) = 1+\frac{3}{(z+2)} - \frac{8}{(z+3)}$$

$$= 1+\frac{3}{z(1+2/z)} - \frac{8}{z(1+3/z)}$$

$$= 1+3/z\,(1+2/z)^{-1} - 8/z\,(1+3/z)^{-1}$$

$$= 1+3/z\left(1-2/z+(2/z)^2-(2/z)^3+\ldots\right) - 8/z\left(1-3/z+(3/z)^2-(3/z)^3+\ldots\right)$$

Laurent's theorem:

Let $f(z)$ be analytic in a region D bounded by $\sum_{n=-\infty}^{\infty} a_n z^n$ concentric circle with c_1 and c_2 with centre a and radius R_1 and R_2 such that $R_1 < R_2$. Let z be any point of D then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^n \text{ where}$$

$$a_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta \text{ and } b_n = \frac{1}{2\pi i} \int_{c_1} f(\zeta)(\zeta-a)^{n-1} d\zeta$$

proof:- Let z be any point in D

Then by Cauchy's integral formula for doubly connected region, we have

$$f(z) = \frac{1}{2\pi i} \left[\int_{c_2} \frac{f(\zeta)}{(\zeta-z)} d\zeta - \int_{c_1} \frac{f(\zeta)}{(\zeta-z)} d\zeta \right] \text{---} (1)$$

Now, consider for first part of eqn (1)

Now for any point c_1 on c_2

$$\frac{1}{\zeta-z} = \frac{1}{(\zeta-a)-(z-a)}$$

$$= \frac{1}{(\zeta-a)\left[1-\frac{z-a}{\zeta-a}\right]}$$

$$= \frac{1}{(\zeta-a)} \left[1 - \frac{z-a}{\zeta-a}\right]^{-1}.$$

By Taylor's series

$$= \frac{1}{\zeta-a} \left[1 + \frac{z-a}{\zeta-a} + \left(\frac{z-a}{\zeta-a}\right)^2 + \dots \left(\frac{z-a}{\zeta-a}\right)^{n-1} + \left(\frac{z-a}{\zeta-a}\right)^n \frac{1}{\left(1-\frac{z-a}{\zeta-a}\right)}\right]$$

$$= \frac{1}{\zeta-a} + \frac{z-a}{(\zeta-a)^2} + \frac{(z-a)^2}{(\zeta-a)^3} + \dots + \frac{(z-a)^{n-1}}{(\zeta-a)^n} + \frac{(z-a)^n}{(\zeta-a)^n(\zeta-z)} \text{---} (2)$$

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta = \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-a)}\,d\zeta + \frac{z-a}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-a)^2}\,d\zeta$$

$$+\dots+\frac{(z-a)^{n-1}}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-a)^n}\,d\zeta+\frac{(z-a)^n}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}\,d\zeta$$

$\mapsto$ (A)

Let $a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta$

(A) becomes,

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-z)}\,d\zeta = a_2+a_1(z-a)+a_2(z-a)^2+\dots+a_n(z-a)^{n-1}+R_n$$

$\mapsto$ (B)

where $R_n = \frac{(z-a)^n}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-a)(\zeta-z)}\,d\zeta$ — (a)

claim:-

$R_n \to 0$ as $n\to\infty$

Let $|z-a| = r \ni: R_1 < r_1 < R_2$ and $|\zeta-a| = R_2$

Consider $|\zeta-z| = |(\zeta-a)-(z-a)|$

$\geq |\zeta-a|-|z-a|$

$= R_2 - r$

$$R_n \leq \frac{r^n}{2\pi}\int_{C_2}\frac{M_1}{R_2^n(R_2-r)}\,|d\zeta|$$

M_1 is the maximum value of $f(\zeta)$ on C_0

from (a)

$$|R_n| \leq \frac{r^n M_1}{2\pi R_2^n(R_2-r)}\int_{C_2}|d\zeta|$$

$$= \frac{r^n M_1}{2\pi R_2^n(R_2-r)}\,2\pi R_2$$

$$= \frac{M_1 R_2}{(R_2-r)}\left(\frac{r}{R_2}\right)^n$$

$\frac{r}{R_2}\to 1$ as $n\to\infty$, $\left(\frac{r}{R_2}\right)^n\to 0$

consequently $R_n\to 0$ as $n\to\infty$

Eqn (3) becomes

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{--- (4)}$$

consider the second part of eqn (1)

let ζ be the point C_1 where $|\zeta-a|=R_1$

consider,

$$\frac{1}{\zeta-a} = \frac{1}{z-\zeta} = \frac{1}{(z-a)-(\zeta-a)} = \frac{1}{(z-a)\left[1-\frac{\zeta-a}{z-a}\right]}$$

$$= \frac{\left[1-\frac{\zeta-a}{z-a}\right]^{-1}}{z-a}$$

$$= \frac{1}{z-a}\left[1+\frac{\zeta-a}{z-a}+\dots+\left(\frac{\zeta-a}{z-a}\right)^{n-1} + \left(\frac{\zeta-a}{z-a}\right)^{n}\left[\frac{1}{1-\frac{\zeta-a}{z-a}}\right]\right]$$

$$= \frac{1}{z-a}+\frac{\zeta-a}{(z-a)^2}+\dots+\frac{(\zeta-a)^{n-1}}{(z-a)^n} + \frac{(\zeta-a)^n}{(z-a)^{n+1}}\cdot\frac{1}{z-\zeta}$$

$$\frac{1}{2\pi i}\int\frac{f(\zeta)}{\zeta-z}\,d\zeta = \frac{(z-a)^{-1}}{2\pi i}\int_{C_1} f(\zeta)\,d\zeta + \frac{(z-a)^{-2}}{2\pi i}\int_{C_1}(\zeta-a)f(\zeta)\,d\zeta + \dots + \frac{(z-a)^{-n}}{2\pi i}\int_{C_1}\frac{(\zeta-a)^n}{(z-\zeta)} f(\zeta)\,d\zeta$$

where $S_n = \dfrac{(z-a)^{-n}}{2\pi i}\displaystyle\int_{C_1}\frac{(\zeta-a)^n}{(z-\zeta)} f(\zeta)\,d\zeta$

let $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1}(\zeta-a)^{n-1} f(\zeta)\,d\zeta$

$$\frac{1}{2\pi i}\int\frac{f(\zeta)}{\zeta-z}\,d\zeta = b_1(z-a)^{-1}+b_2(z-a)^{-2}+\dots+b_n(z-a)^{-(n-1)} + S_n \rightarrow (5)$$

claim $S_n\to 0$ as $n\to\infty$

For let $|z-a|=r$ and $|\zeta-a|=R_1$

$$|z-\zeta| = |(z-a)-(\zeta-a)|$$
$$\geq |z-a|-|\zeta-a|$$
$$= r-R_1$$

$$|S_n| \le \frac{r^n}{2\pi} \frac{M_2 R_1^n}{r - R_1} \int_{C_1} |d\xi|$$

$$|S_n| \le \frac{r^n}{2\pi} \frac{M_2 R_1^n}{r - R_1} 2\pi R_1$$

$$= \frac{M_2 R_1}{r - R_1} \left(\frac{R_1}{r}\right)^n$$

$\therefore \left(\frac{R_1}{r}\right) < r$ we have $\left(\frac{R_1}{r}\right) \to 0$ as $n \to \infty$

consequently $S_n \to 0$ as $n \to \infty$

Eqn (5) becomes

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=1}^{\infty} b_n (z-a)^n \quad \text{— (6)}$$

Eqn (4) and (6) sub

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^n$$

where $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi$ and

$$b_n = \frac{1}{2\pi i} \int_{C_1} (\xi - a)^{n-1} f(\xi) \, d\xi$$

Hence proved.

1) Find the Laurent's series expansion of $f(z) = z^2 e^{1/z}$ about $z = 0$.

Soln: Given that $f(z) = z^2 e^{1/z}$

$$f(z) = z^2 \left[1 + (1/z) \tfrac{1}{1!} + (1/z)^2 \left(\tfrac{1}{2!}\right) + \cdots\right]$$

$$= z^2 \left(1 + 1/z + \tfrac{1}{z^2 2!} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$

2) Expand $\frac{1}{z(z-1)}$ as Laurent's series i) about $z=0$ in power of z and ii) about $z=1$ in power z^{-1}, Also it in the region of validity.

Ans: The only points where $F(z)$ is not analytic are 0 and 1.

Hence $f(z)$ can be expressed as a Laurent's series in the $0<z<1$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z(-1)(1-z)} = -\frac{1}{z}(1-z)^{-1}$$

$$f(z) = -\frac{1}{z}(1+z+z^2+z^3+\dots)$$

since $|z|<1$

$$f(z) = -\left[\frac{1}{z}+1+z+z^2+z^3+\dots\right]$$

This is Laurent series expansion $f(z)$ is $0<|z|<1$

ie) $f(z)$ is analytic in $0<|z-1|<1$

Hence can be expanded as a Laurent's series in power of $z-1=0$ in the region

$$f(z) = \frac{1}{z(z-1)} = \left(\frac{1}{z-1}\right)\left(\frac{1}{1-1+z}\right)$$

$$f(z) = \frac{1}{z-1}\left(\frac{1}{1+(z-1)}\right) = \frac{1}{z-1}(1+(z-1))^{-1}$$

$$f(z) = \left(\frac{1}{z-1}\right)[1+(z-1)+(z-1)^2+\dots]$$

Since $|z-1|<1$

$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \dots$$

This gives the Laurent's series expansion in $0<|z-1|<1$

5m **Thm** $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $\sum_{n=1}^{\infty}\log(1+a_n)$ converges.

proof: Let $S_n = \sum_{k=1}^{n}\log(1+a_k)$

$$P_n = \sum_{k=1}^{n}\log(1+a_k) = \prod_{k=1}^{n} P_k$$

If $\sum_{n=1}^{\infty} \log(1+a_n)$ converges then $s_n \to s$ for some $s \in C$,

e^s is continuous, $e^{s_n} \to e^s$ as $n \to \infty$

$$e^{s_n} = e^{\sum_{k=1}^{n} \log(1+a_k)} = \prod_{k=1}^{n} e^{\log(1+a_k)}$$

$$= \prod_{k=1}^{n} (1+a_k) = P_n$$

$$\Rightarrow P_n \to e^{s_n} \text{ as } n \to \infty$$

$$\Rightarrow \prod_{n=1}^{\infty} (1+a_n) \text{ is converges}$$

conversely,

Assume that $\prod_{n=1}^{\infty} (1+a_n)$ converges

$\Rightarrow P_n \to P$ as $n \to \infty$ for some $P \in C$

For each n there exists $h_n \in z$ such that

$$\log\left(\frac{P_n}{P}\right) = s_n - \log P + 2\pi i h_n \to (1)$$

where h_n is such that $\cancel{\log}$ $\left|\arg\left(\frac{P_n}{P}\right)\right| < \pi$

We claim that h_n is constant after some stage

Now, $\log\left(\frac{P_{n+1}}{P}\right) = s_{n+1} - \log P + 2\pi i h_{n+1} \to (2)$

(2) - (1)

$$\Rightarrow \log\left(\frac{P_{n+1}}{P}\right) - \log\left(\frac{P_n}{P}\right) = s_{n+1} - s_n + 2\pi i (h_{n+1} - h_n).$$

$$\log\left(\frac{P_{n+1}}{P}\right) - \log\left(\frac{P_n}{P}\right) - (s_{n+1} - s_n) = 2\pi i (h_{n+1} - h_n)$$

Taking arg,

$$(h_{n+1} - h_n) 2\pi = \arg\left(\frac{P_{n+1}}{P}\right) - \arg\left(\frac{P_n}{P}\right)$$

$$- \arg(1+a_{n+1})$$

$$(h_{n+1} - h_n) 2\pi = \arg\left(\frac{P_{n+1}}{P}\right) - \arg\left(\frac{P_n}{P}\right) - 0 \text{ as } n \to \infty$$

$\Rightarrow h_{n+1} = h \;\forall\; n = m$ for some $m \in N$

$(\because P_{n+1} \to P,\ P_n \to P)$

$$S_n = \log\left(\frac{P_n}{P}\right) + \log P = h\,2\pi i$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \log\left(\frac{P_n}{P}\right) + \log P = h\,2\pi i$$

$$= \log P - h\,2\pi i$$

$\therefore \sum_{n=1}^{\infty} \log(1+a_n)$ converges.